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Digitization of partitions and of tessellations

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Outline of the lecture

1. **Regular open sets**
2. Tessellations
3. Digital Tessellations by Khalimsky topology
4. Digital Tessellations by Voronoi polyhedra
5. Conclusion

Purpose

When one partitions a set E the frontiers between classes are not materialized ...But some operations need them.

- How to combine interiors and frontiers in a unique representation?
- Is that possible with any partition of E?

  or is there some condition for the classes?

... Here are the problems we try to solve.

Continuous or Digital Space

One usually separates the questions specific to $\mathbf{R}^n$ or to $\mathbf{Z}^n$.

But the notions involved in our problem are prior to this distinction...It is wiser not to specify it too soon, and to work on some space E.

Moreover, we would like to find bridges between $\mathbf{R}^n$ and $\mathbf{Z}^n$.

Structure of P(E)

The inclusion relation

$$X \subseteq Y \quad \Leftrightarrow \quad \{x \in X\} \Rightarrow \{x \in Y\} \qquad X, Y \in \mathcal{P}(E)$$

is an ordering, i.e.

- $X \subseteq X$
- $X \subseteq Y$ and $Y \subseteq X \quad \Rightarrow \quad Y = X$
- $X \subseteq Y$ and $Y \subseteq Z \quad \Rightarrow \quad X \subseteq Z$

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This ordering induces a complete lattice where any family $\{X_j \quad j \in J\}$ admits

- smaller upper bound $\cup X_j$
- largest lower bound $\cap X_j$

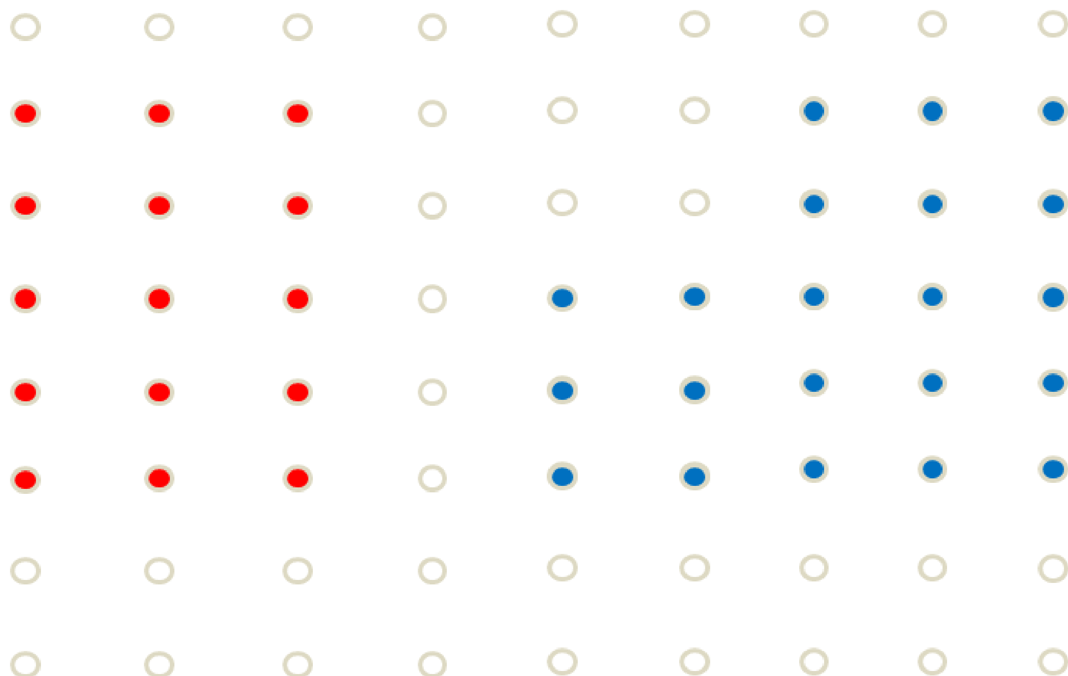
Open and closed sets of P(E)

- Povide a topology or a metric with E and consider $B \in \mathcal{P}(E)$
- Set $B = \overline{B}$ is closed when it contains its fontiers ;

  Set $B = B^{\circ}$ is open when it does not
- The complement of the interior $B = B^{\circ}$ is the adherence of the complement of B
- In case of a metric E, B° open $\sim B = \cup\{\text{all open discs} \subseteq B\}$

  (Example in $\mathbf{Z}^2$ with unit disc the 3×3 square)

Open and closed sets of P(E)



Both blue and red sets are regular open and regular closed for the topology of the square metric

Regular open sets

An open set B is regular, or $\mathcal{R}$-open, when $B = (\overline{B})^\circ$

$\mathcal{R}$ stands for the family of the regular sets B of $\mathcal{P}(E)$.

Regular open sets

An open set B is regular, or $\mathcal{R}$-open, when $B = (\overline{B})^\circ$

$\mathcal{R}$ stands for the family of the regular sets B of $\mathcal{P}(E)$.

$\mathcal{R}$ is a complete lattice for the inclusion ordering, where the supremum and the infimum are given by

$$\vee B_i = (\overline{\cup B_i})^\circ \quad ; \quad \wedge B_i = (\cap B_i)^\circ .$$

and the unique complement of B is

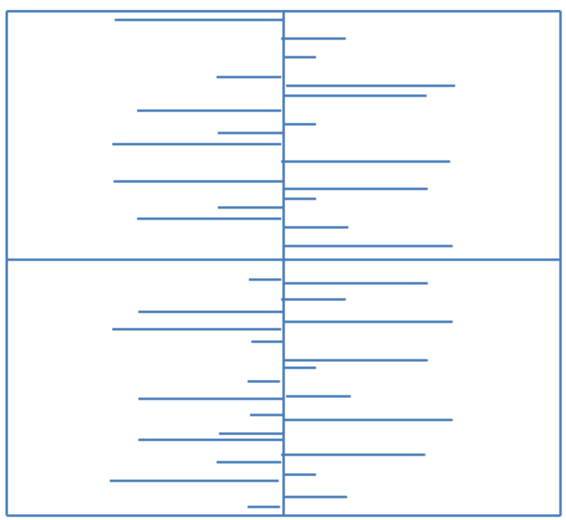
$$(compB)^\circ$$

Regular open sets

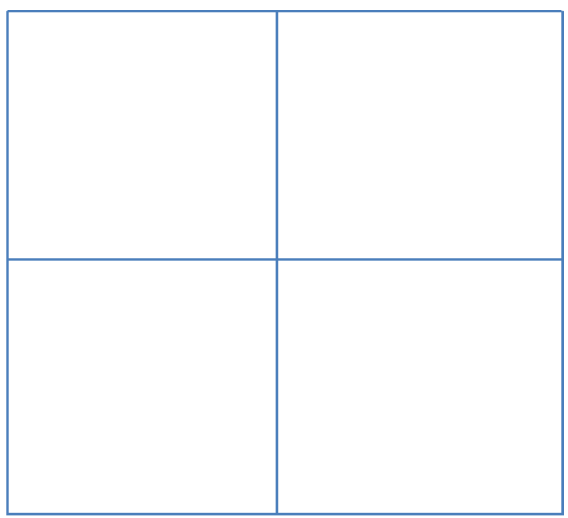
Denote by $S = (\overline{B})^\circ$ the $\mathcal{R}$-open transform of the open set $B \in \mathcal{G}(E)$.

The operation $B \to S = (\overline{B})^\circ$ is an **algebraic closing** on the open sets of E, and the image of $\mathcal{G}$ is $\mathcal{R}$.
This closing means that S is the smallest $\mathcal{R}$-open set that contains B.

For example, if we take Fig.a for B, then we obtain Fig.b for transform S.



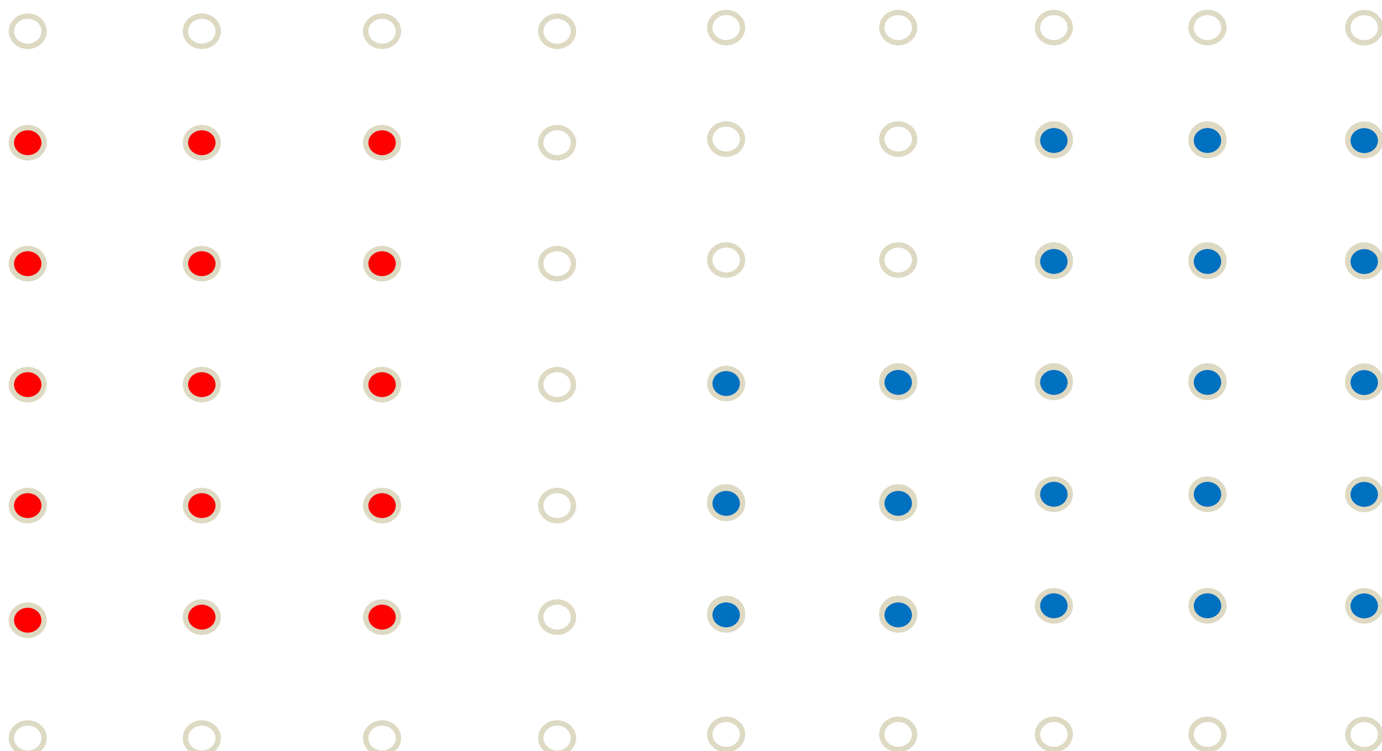
a



b

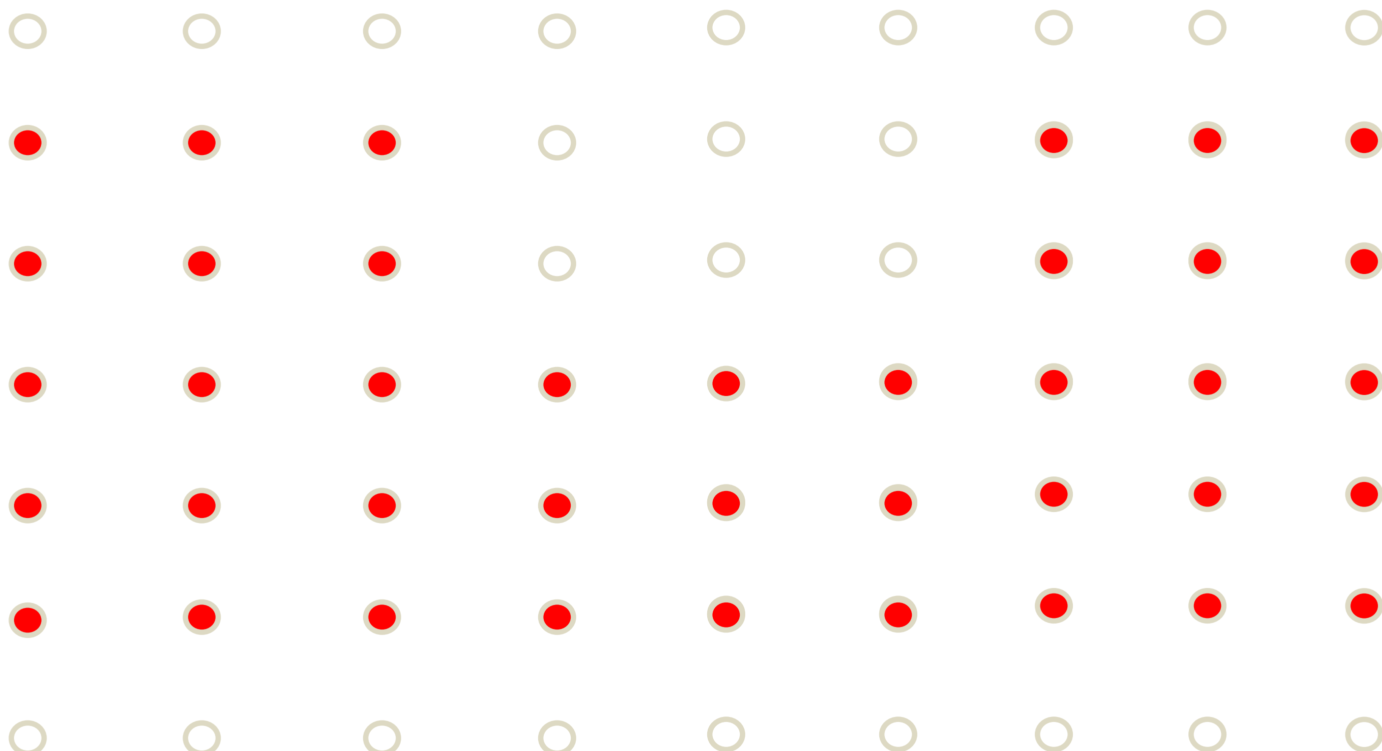
Regularisation

Taken individually, both sets are open and regular for the square metric but their union, still open, is no longer regular.



Regularisation

We regularize by the **algebraic closing** $B \to S = (\overline{B})^\circ$
This closing means that S is the smallest $\mathcal{R}$-open set that contains B.



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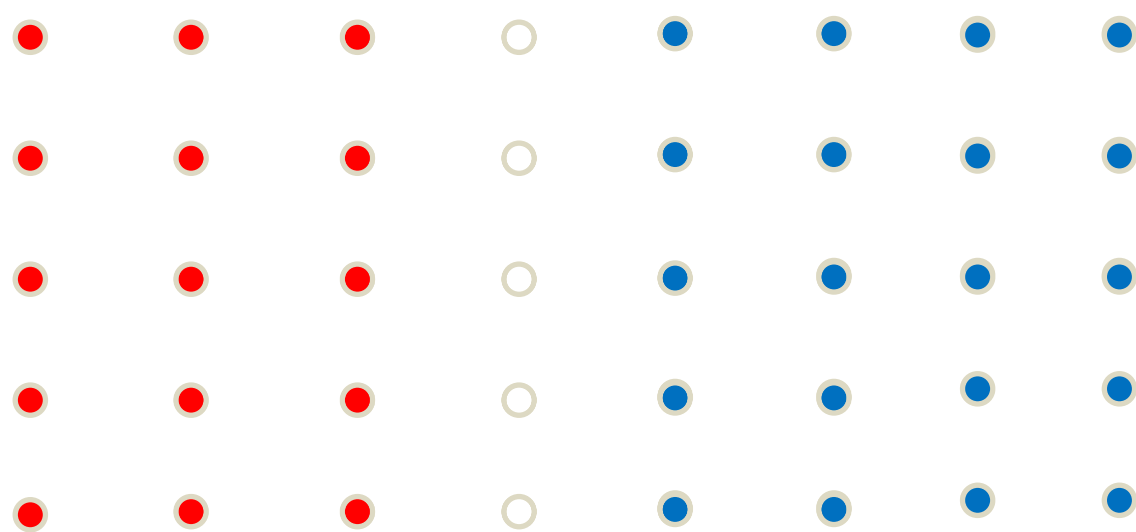
Tessellation

A tessellation τ of E is a family of disjoint open *classes* $\{B_i, i \in I\}$,

$$\tau = \{B_i, i \in I\} \text{ with } i \neq j \Rightarrow B_i \cap B_j = \phi$$

whose union $\cup B_i$ plus the union of all frontiers $Fr(B_i, B_j), i \neq j$ partitions space E.



Tessellation an regular sets

Theorem: The family $\tau = \{B_i, i \in I\}$ is a tessellation if and only if all B_i are $\mathcal{R}$-open.

The theorem reminds us of Jordan's one, though

- it is true in any topological space,
- it does not focus on the frontiers, but on the classes.
- connectivity is not involved, whereas it is essential in Jordan's one

In $\mathbf{R}^2$ every Jordan curve induces a tesselation , but a tesselation into two open classes, even connected, can have a frontier which is not a Jordan curve.

Here is a digital contour which separates $\mathcal{R}$-open sets and which is thick:

Hierarchies of tessellations

- The tessellations met in image processing are often associated with hierarchies, i.e. are elements of totally ordered closed families.
- The classes $\{s_i\}$ of the minimal tessellation τ_0 are called "the leaves", and are supposed in locally finite number.
- These leaves are indivisible $\mathcal{R}$-open sets, i.e. each class of a larger tesselation contains one leave at least and is disjoint from those that it does not contain.
- The set E itself, considered as a class (and which is $\mathcal{R}$-open), ends the hierarchy.

Hierarchies of tessellations

In a hierarchy, the classes of a tessellation τ do not reduce to union of their leaves: the portions of frontiers between adjacent leaves would belong to no classe

We must find out a law of composition

Let us partition the totality of the leaves into sub-sets

$$B_j = \cup\{s_j, j \in J\}$$

Then the unique tessellation which keeps disjoints the B_j clusters has for classes the $\mathcal{R}$-open sets $S_j = (\overline{B_j})^{\circ}$.

Hierarchies of tessellations

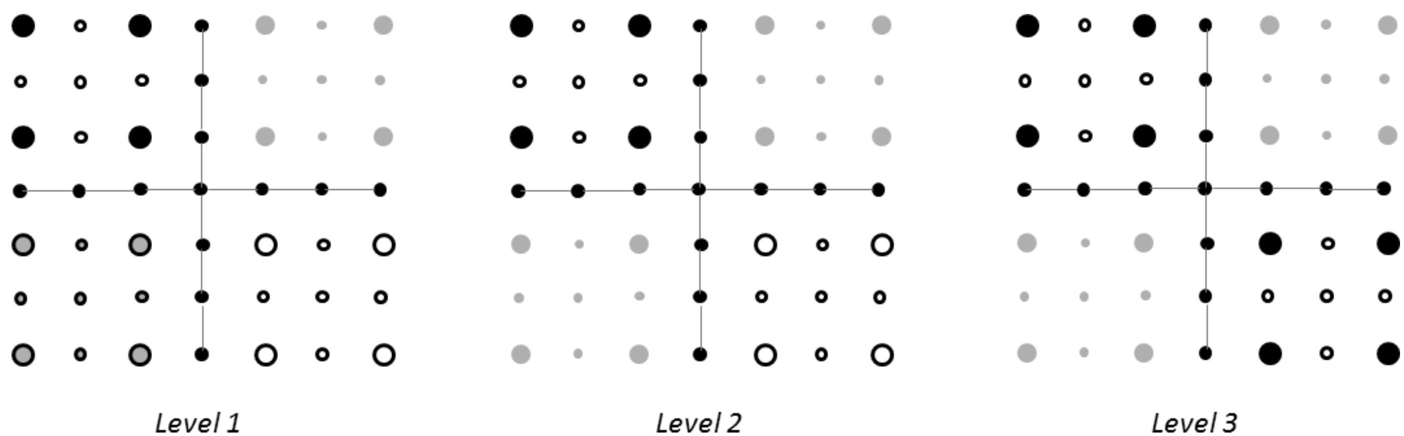
Theorem: The set $\mathcal{T}$ of all tessellations $\tau \geq \tau_0$ of E forms a complete lattice for the ordering of the regular sets.

- Its universal elements are τ_0 and E.
- The infimum of family $\{\tau_p, p \in P,\ \tau_p \geq \tau_0\}$ is the tesselation whose class at point x is the infimum, in $\mathcal{R}$, of the classes of the τ_p at point x,
- and the supremum is the smallest tessellation whose classes are suprema of the classes of the τ_p in $\mathcal{R}$.

The connectivity trouble

Up to now, no condition on connectivity was introduced.

Hierarchies of tessellations do not need it. But on the other hand, they do not preserve it:



Connected sets will merge into connected sets iff the ambiguous configurations do not exist, i.e. iff the adherences of the classes never intersect by a point (in 2-D) or a segment (in 3-D).

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Reminder on Khalimsky topology

Associate:

- the open interval]m-(1/2),m'+(1/2)[with every pair m=m' of odd integers, and
- the closed interval [n-(1/2),n'+(1/2)] with every pair n=n'of even integers.

The unions of open (resp. closed) intervals generate a non separated topology.

Reminder on Khalimsky topology

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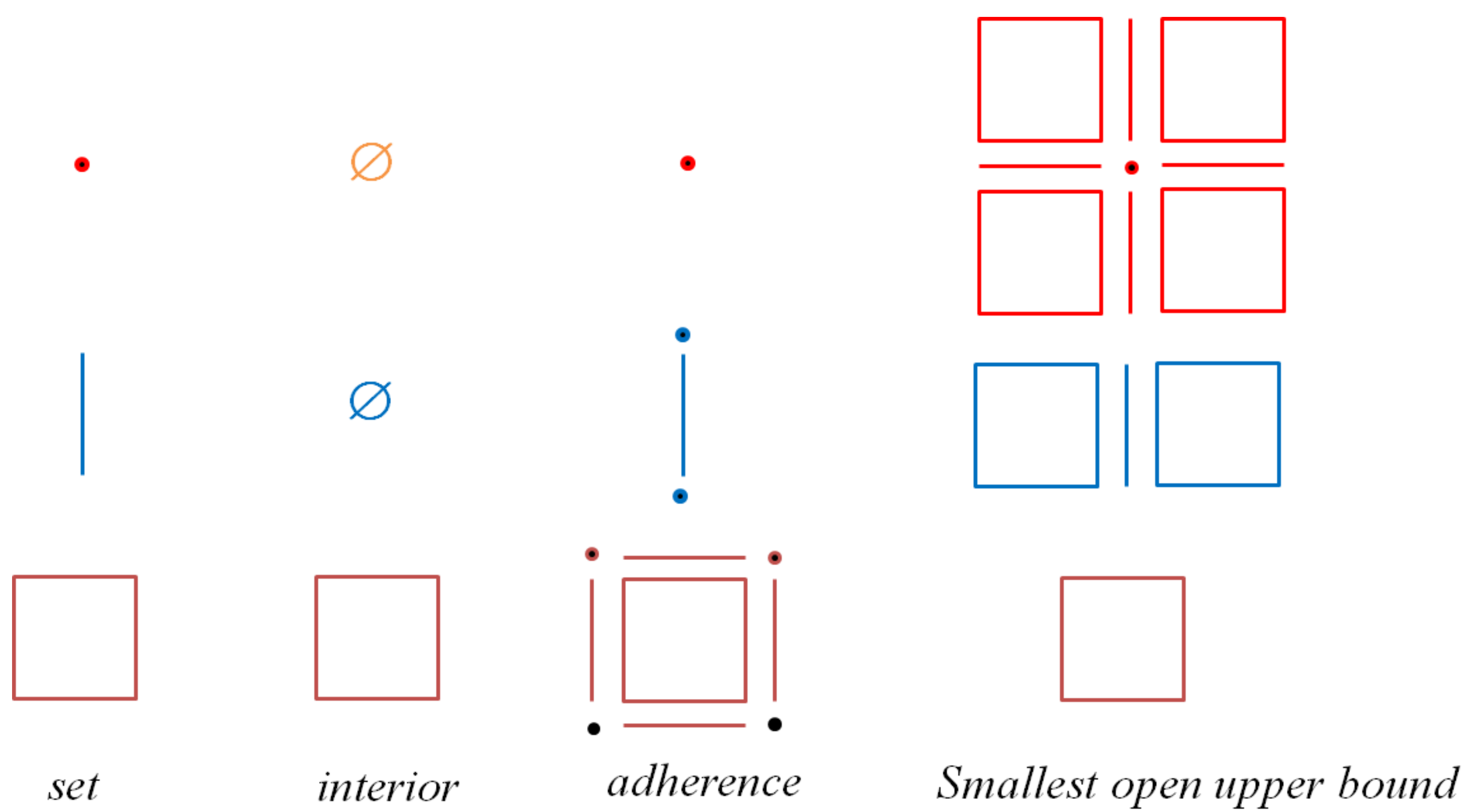
The unions of open (resp. closed) intervals generate a non separated topology.

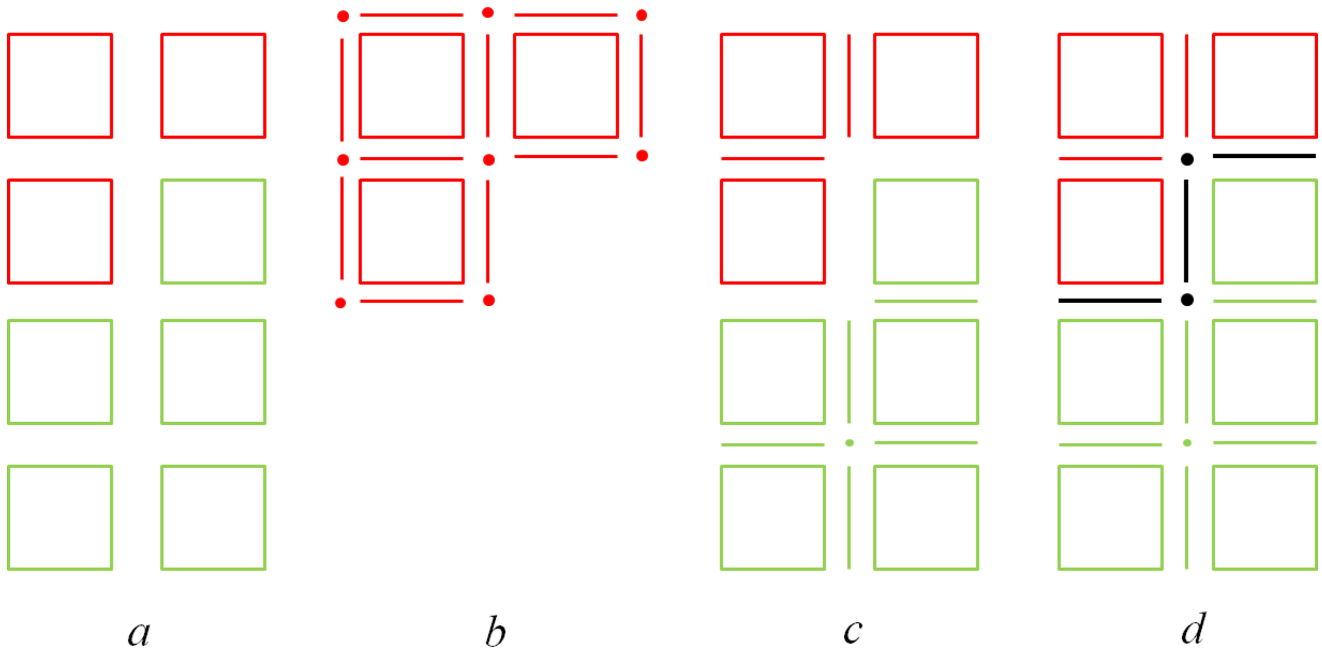
The passage to $\mathbf{R}^n$ is obtained by product topology of n Khalimsky lines, where

- The n-cubes whose all coordinates of the centers are odd are open, those with all coordinates are even are closed;
- the others cubes are "mixed".

Khalimsky Rules



Kovalevsky cells



In $\mathbf{R}^2$ the Kovalevsky cells display Khalimsky topology. The Figure shows an example with (a) eight open elementary cells, (b) one elementary close cell (c) two R-open sets, and (d) their frontier.

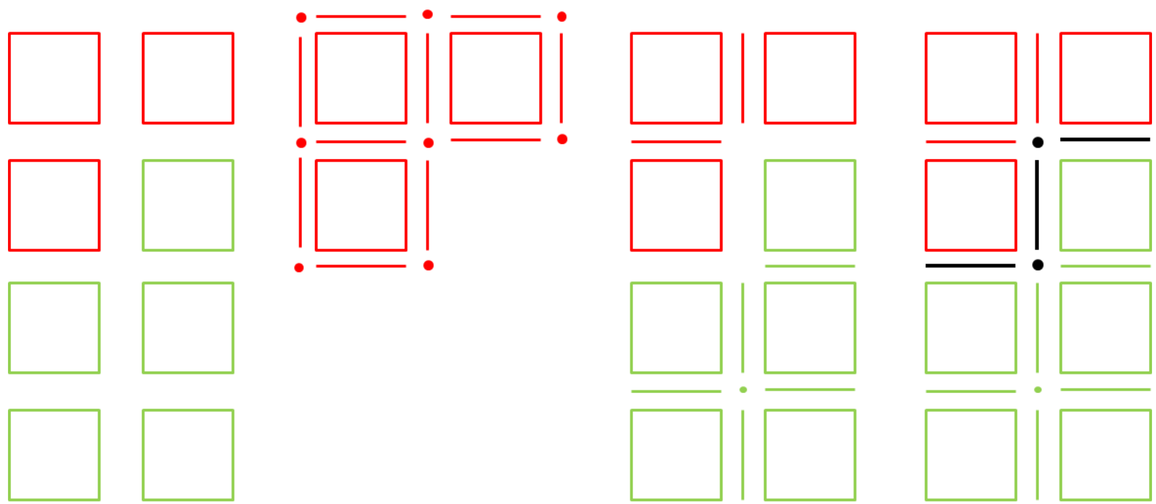
This structure is akin to simplicial simplexes.

Doubling $\mathbf{Z}^2$

Interpret the points of a set $X \subseteq \mathbf{Z}^2$, as points of odd coordinates in a Khalimsky plane $\mathbf{K}^2$ which contains twice more points by line and twice more lines.

In the double resolution plane $\mathbf{K}^2$, all points of the background have odd coordinates.

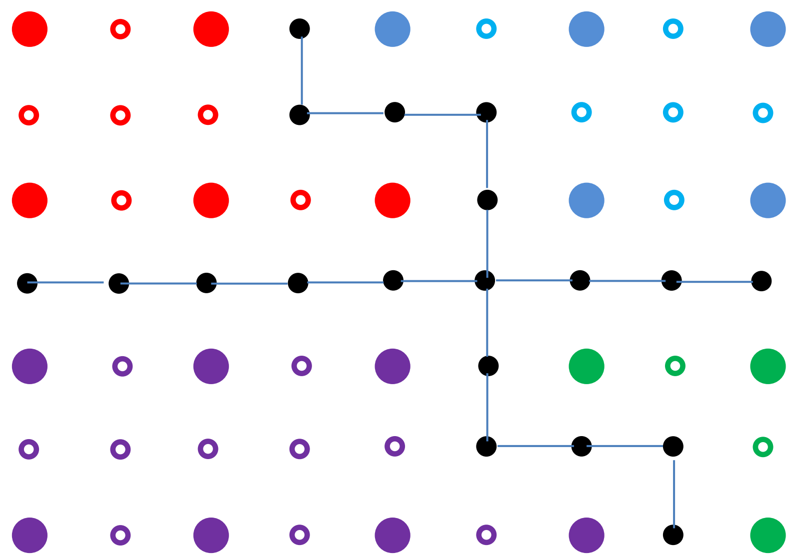
We meet the classical rule of the double sampling : The fine mesh displays the net of the closed contours which envelop the open classes.



a *b* *c* *d*

Doubling Z^2

For the sake of simplicity we will indicate the previous Kovalevky squares by big dots, and the additional segments and points by small dots.



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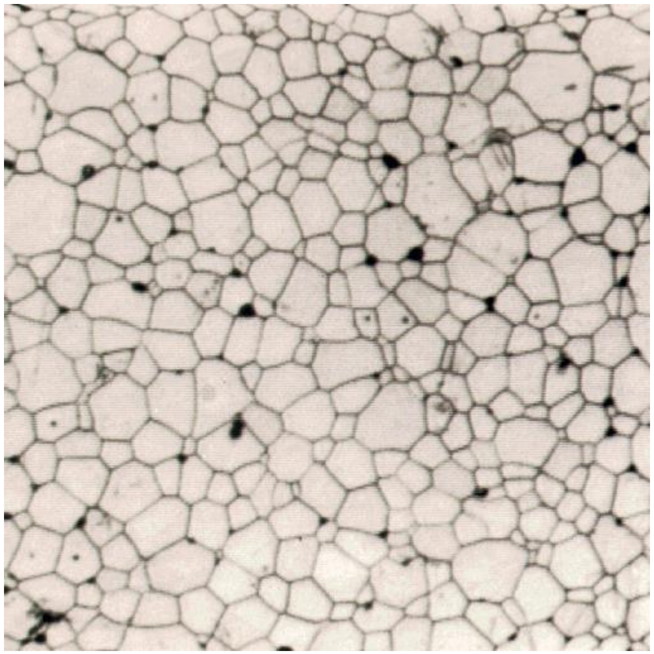
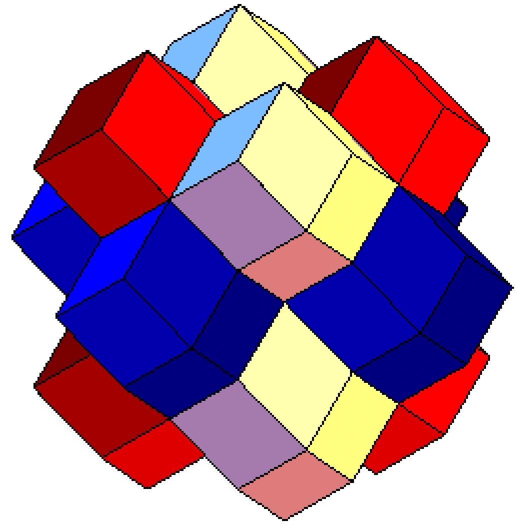
Voronoi polyhedra

Let $X \subseteq \mathbf{R}^n$ be a locally finite set of *centres*.

- Associate with each centre $x \in X$ the so-called Voronoi polyhedron $Q(x)$ of all points $y \in \mathbf{R}^n$ closer to x than to any other centre.
- $Q(x)$ is convex and open, hence regular, so that the set $\{Q(x), x \in X\}$ of all Voronoi generates a tessellation of $\mathbf{R}^n$.
- In particular in $\mathbf{R}$, when the centres are the points m of odd integer abscissae m, the corresponding Voronoi is $]m - \frac{1}{2}, m + \frac{1}{2}[$, we find again Khalimsky topology.

Voronoi polygons and polyhedra





Two examples, one from physics (cristallography) the other by simulation

Conditions for Voronoi polyhedra

Impose the following two conditions to the Voronoi polyhedra in $\mathbf{R}^n$:

1. *they must be identical, up to a translation (i.e. regular grid);*
2. *the adherences of two adjacent polyhedra always have a common face of n-1 dimension.*

First condition admits

- only two solutions in $\mathbf{R}^2$, the square and the hexagon,
- and five in $\mathbf{R}^3$, the cube, the hexagonal prism, the truncated octahedron, and the two elongated and rhombic dodecahedra.

Voronoi Polyhedra and Connectivity

The second condition ensures that connected sets merge into connected sets (the ambiguous configurations do not exist)

It reduces the possibilities to the only hexagon in 2-D, and only truncated octahedron in 3-D.

In 2-D the centres describe the triangular grid, in the 3-D the centred cubic grid

If we are not interested in translation invariance (resp. in preserving connectivity), the first (resp. the second) condition becomes cumbersome,

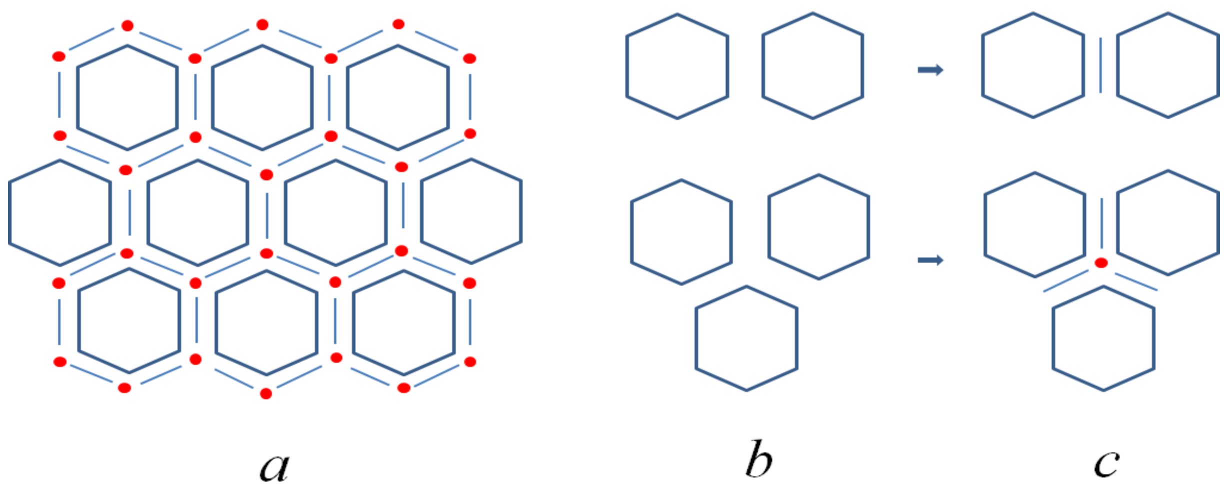
Hexagonal Voronoi topology in $\mathbf{R}^2$

$\mathbf{R}^2$ is repaired by three axes of coordinates at $120°$, and the origin $(1, 1, 1)$.

- Take for centres all points of the plane whose coordinates are odd on each of the three axes.
- The associated Voronoi polygons are the hexagons.
- They generate the **hexagonal topology** which is no longer Khaminski excepted in $\mathbf{R}^1$

Hexagonal tessellation of $\mathbf{R}^2$

- The grid is triangular.
- The Voronoi polygons are open hexagons.
- The other open sets are the unions of these hexagons plus the edges adjacent between them,
- and the triple points are closed

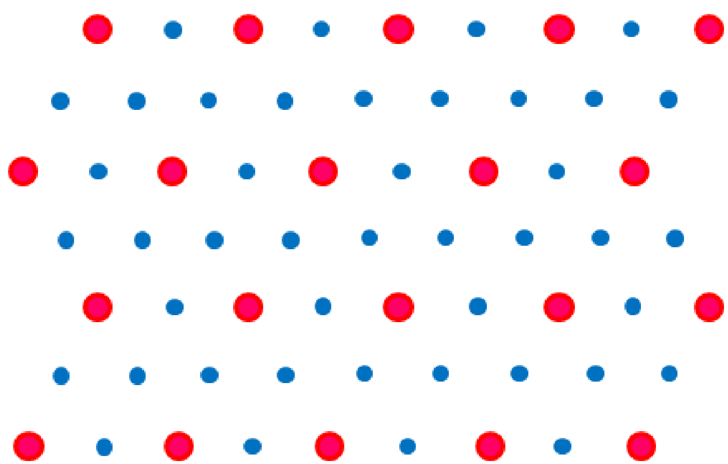


Doubling $\mathbf{Z}^2$

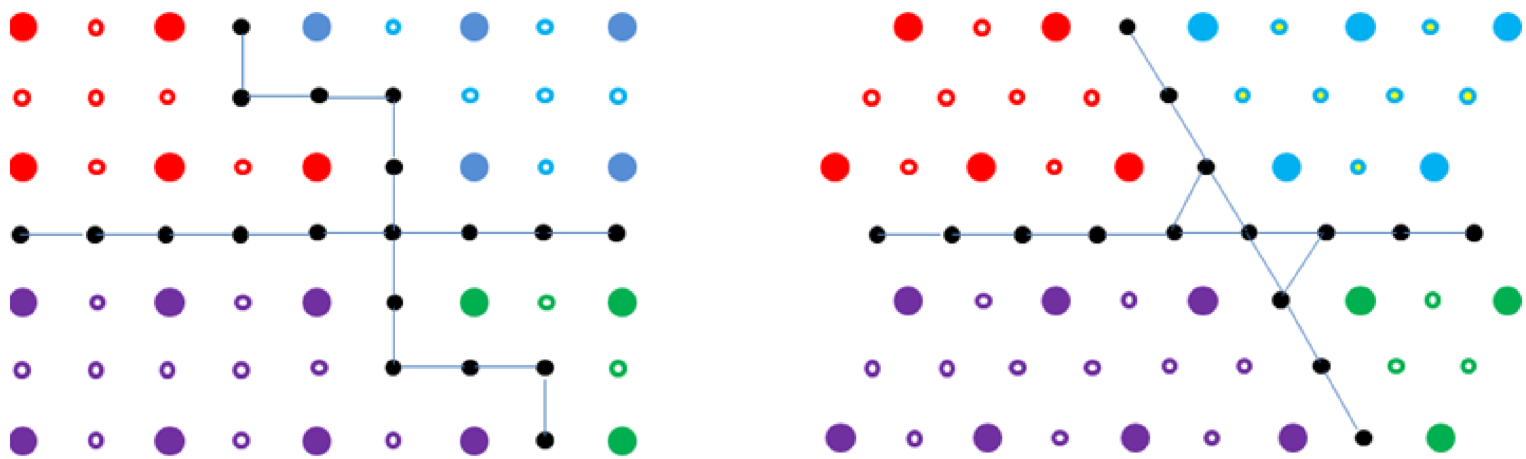
$\mathbf{Z}^2$ is the triangular grid, and one doubles the pixels along all lines parallel to the three axes $\Rightarrow$ new space $\mathbf{H}^2$.

The points of $\mathbf{Z}^2$ have three odd coordinates in $\mathbf{H}^2$, and are identified to the open sets of the hexagonal topology of $\mathbf{R}^2$, the other points of $\mathbf{H}^2$ being closed.

In this new topological space, the $\mathcal{R}$-open version $(\overline{X})^\circ$ of $X \subseteq \mathbf{Z}^2$ is obtained by adding to X all points comprised between each two open points de X in each of the three directions.



Comparison with Khaminsky topology

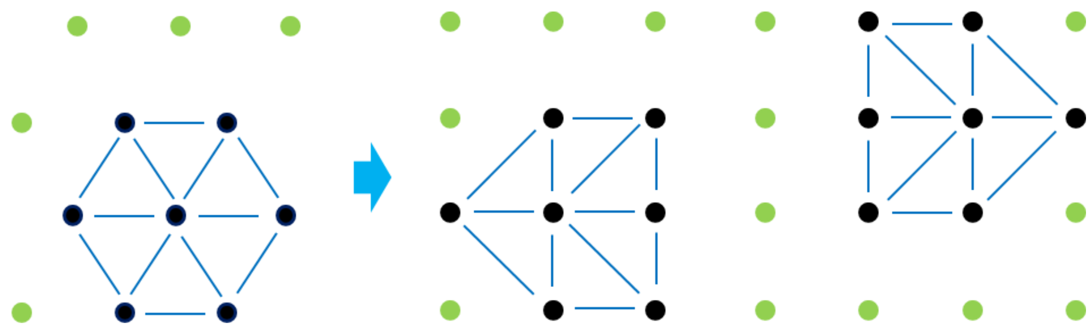


The small black points indicate the net of the frontiers of the tessellation in $\mathbf{H}^2$.

Two major differences with Khalimsky square grid.

- the frontiers are no longer simple arcs (clusters of pixels may appear),
- no ambiguous diagonal were removed by suppression of the quadruples points.

Hexagonal emulation

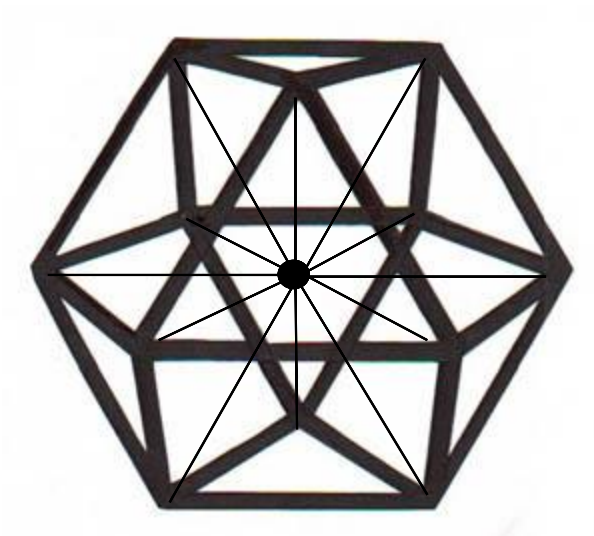


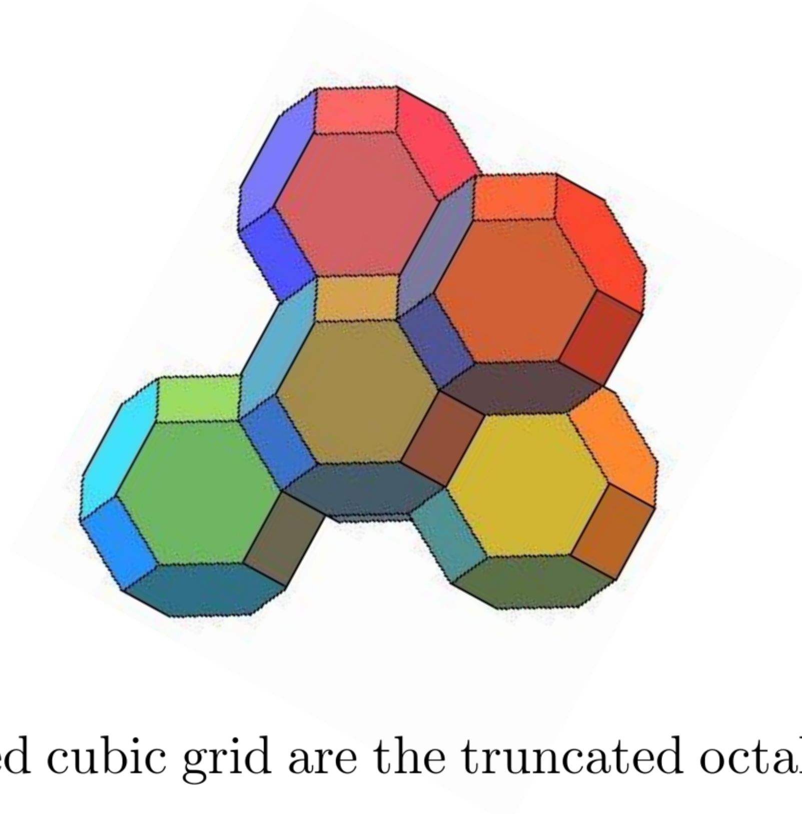
Hexagonal grid can be emulated from a square grid.

the shape of the elementary hexagon depends on the parity of its central line, but this irregularity is rapidly absorbed with the sizes of the objects.

This grid is self dual, i.e. the connectivity is not changed when one takes the negative of a set, and there is no longer consistency trouble in the saliency functions.

Tessellation of $\mathbf{R}^3$ by truncated octahedra

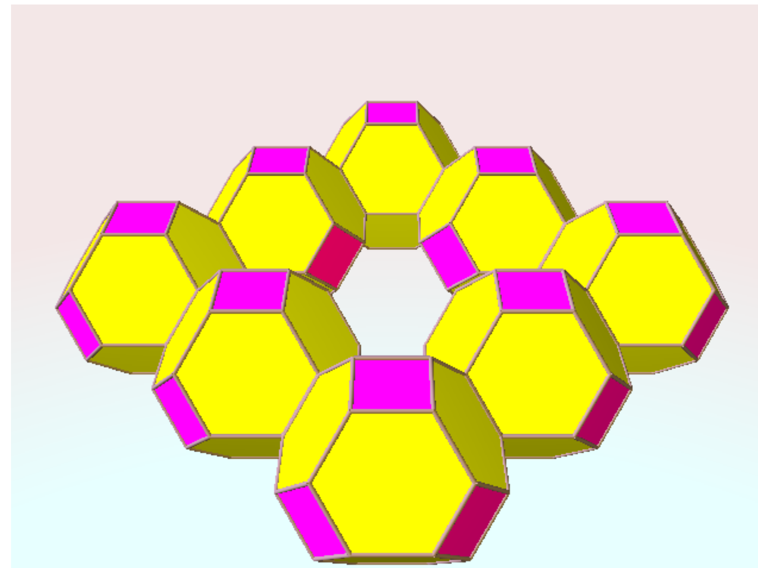




The Voronoi polyhedra of the centred cubic grid are the truncated octahedra -or tetrakaidecahedra-

They partition $\mathbf{R}^3$ in open polyhedra, square and hexagonal faces, triple edges and quadruple vertices. These elements generate a digital topology.

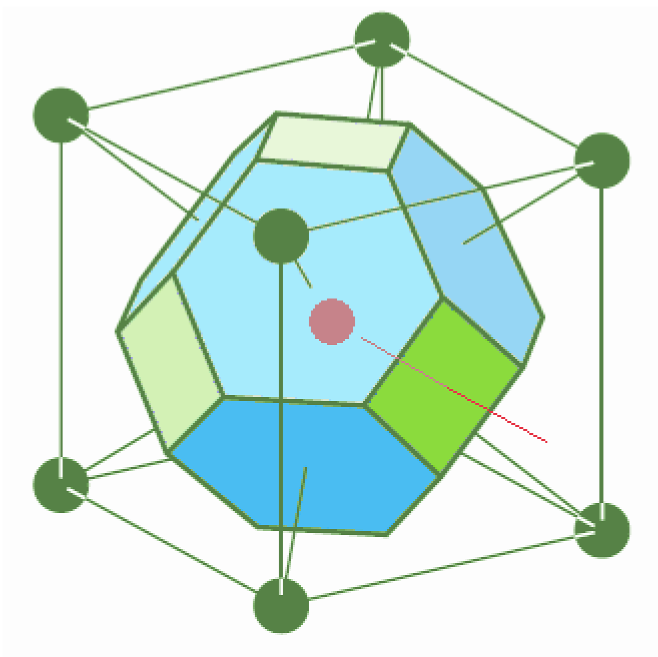
Tessellation of $\mathbb{R}^3$ by truncated octahedra

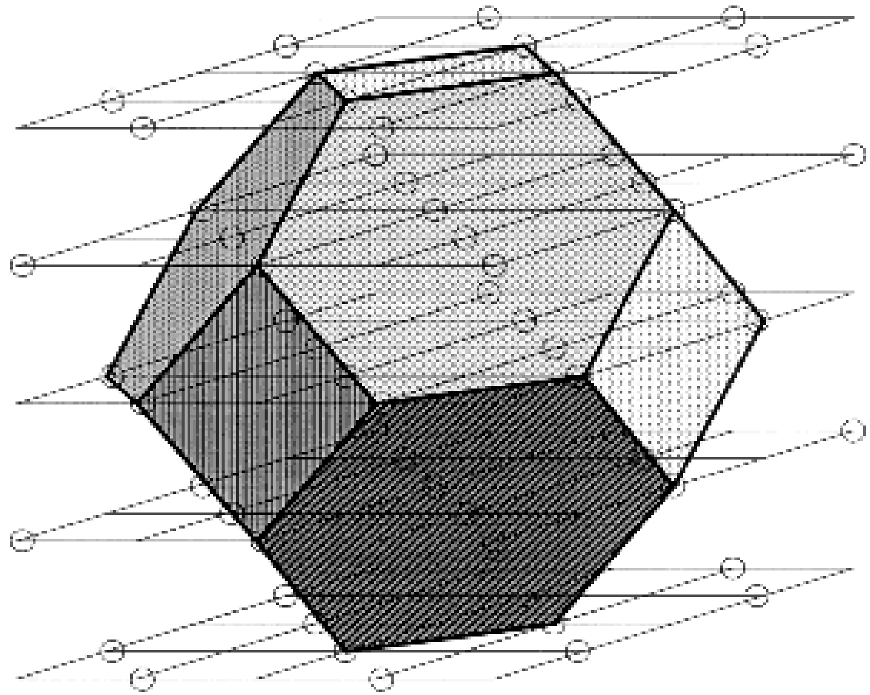


The regularization fills up the internal 1-D or 2-D fissures of zero thickness, and the background net is a connected union of faces and edges which completely envelops the classes.

Two adjacent truncated octahedra always share a face (which is not the case with the cubes)

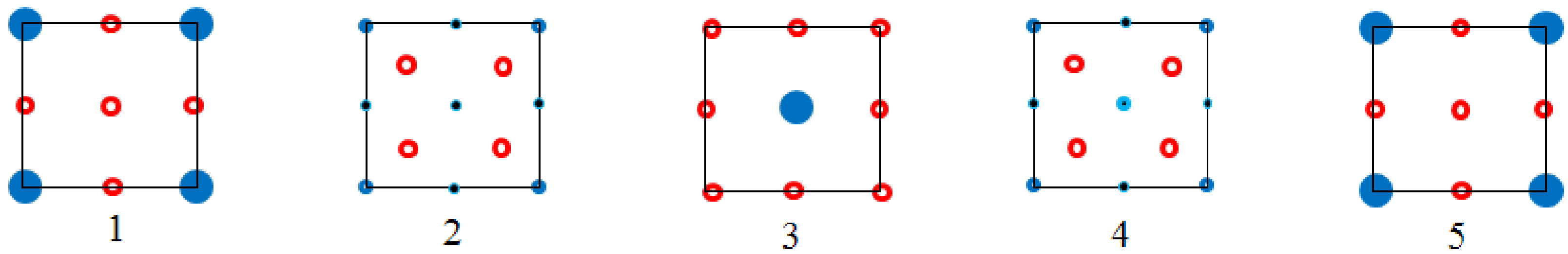
Tessellation of Z^3 by truncated octahedra





The unit digital truncated octahedron requires five sections.

Tessellation of Z^3 by truncated octahedra



Points of the initial truncated-octahedron of $\boldsymbol{Z}^3$

Points added by half spacing in seven directions (sides and diagonals of the cube)

Points added to regularize t he truncated-octahedron

For the rule $\mathbf{Z}^3 \to \mathbf{H}^3$ one starts from three horizontal planes of the cubic grid containing the vertices (n° 1 and 5) and the centre (n° 3) of the unit cube . The planes n° 2 and 4 are added for generating a centred cubic grid twice finer.

In the three directions of the cube and the four ones of the main diagonals alternate points of $\mathbf{Z}^3$ with those added for forming $\mathbf{H}^3$.

Tessellation of Z^3 by truncated octahedra

Comments

- The structure reminds that of the triangular grid of $\mathbf{Z}^2$ and the passage $\mathbf{Z}^2 \to \mathbf{H}^2$.
- Again the tesselation reduces the cells to the two types of the (open) truncated octahedra, and the (closed) square or hexagonal faces, i.e. something that can be described in terms of graphs.
- The centred cubic grid can easily be emulated by shifting horizontally the even planes by the vector $(1, 1, 0)$

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Conclusion

- We proposed a method to combine interiors and frontiers in a unique representation
- The problem was solved by means of regular open sets, and further, tessellations
- In digital cases the passage partition$\rightarrow$tessellation involves double resolution
- The ambiguous configurations of the quare and cubic grids are soved when they are replaced by hexagonal ($\mathbf{Z}^2$) or tetrakaidecahedral grids ($\mathbf{Z}^3$).

It was shown in detail how tessellation, Alexandrov topology, and double resolution interfere. In practice, it is suggested to favour in the triangular grids in 2-D and the centred-cubic ones in 3-D.

Thanks

For your

attention !

References

[1] B. R. Kiran, J. Serra, Digitalisation de partitions et de tessellations, HAL-01137354v1 march 2014

[2] J. Serra, Digitalisation of partitions and of tessellations, (submitted to DGCI 2016)

References

[1] P. Alexandrov, Diskrete Raüme. *Mat.Sb.* **2** (44),. 501-519, 1937

[2] G. Bertrand, Completions and Simplicial Complexes, in *Discrete Geometry for Computer Imagery*, LNCS 6007 Springer,129-140, 2011

[3] J. Cousty, G. Bertrand, M. Couprie, L. Najman, Collapses and watersheds in pseudomanifolds of arbitrary dimension, *JMIV*, 50 (3),.261-285, 2014.

[4] H.J.A.M. Heijmans, Morphological image operators. *Advances in Electronics and Electron Physics,* suppl. 24, Vol. 50, Ac. Press,1994.

[5] E. Khalimsky Topological structures in computer sciences *J. Appl. Math. Simulation*, 1(1), 25-40, 1987

[6] B. R. Kiran, J. Serra, Fusions of Ground Truths and of Hierarchies of Segmentations, *Pattern Recognition Letters* 47 63–71, 2014

References

[1] V.A. Kovalevsky, Finite topology as applied to image analysis. *Comput. Vision Graph. Image Process.* 46: 141-161 1989

[2] G. Matheron, *Eléments pour une théorie des mileux poreux*, Masson, Paris, 1969.

[3] G. Matheron, *les treillis compacts* Tech. Report, Ecole des Mines de Paris, 183p.,1996

[4] L. Mazo, N. Passat, M. Couprie, C. Ronse. Paths, homotopy and reduction in digital images. *Acta Applicanda Mathematicae,*, 113 (2),.167-193, 2011

[5] E. Melin, Digital Surfaces and Boundaries in Khalimsky Spaces, *J. Math Imaging Vis.*, 28 , 169-177, 2007.

[6] Ch. Ronse, $\mathcal{R}$-open or closed sets, WD58, Philips Research Lab, 1990

[7] J. Serra, Cube, Cube-octahedron or Rhombododecahedron as Bases for 3-D shape Descriptions, *Advances in Visual Form Analysis*, World Scientific 502-519, 1997.