Recursive structure of digital planes, a combinatorial approach based on continued fractions

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Outline

1. Recursive Structure of Digital line
2. Construction guided by Euclid
3. Generalization to higher dimensions

Part I

Recursive Structure of Digital line

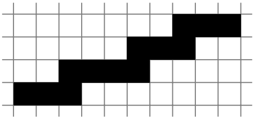
1. Definition
2. Periodic structure
3. Christoffel words
4. Digital convexity test

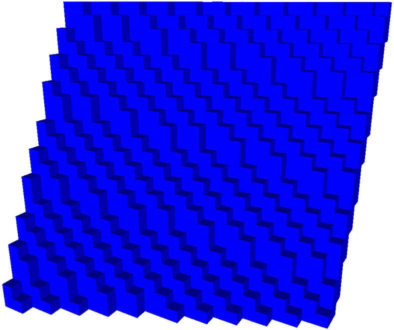
Digital lines and planes

Definition ([Reveillès 91])

The digital hyperplane $\mathcal{P}(v,\mu)$ with **normal vector** $v \in \mathbb{Z}^d$, **shift** $\mu \in \mathbb{R}$ is the subset of $\mathbb{Z}^d$ defined by:

$$\mathcal{P}(v,\mu) = \left\{ x \in \mathbb{Z}^d \mid \mu \leq \langle x, v \rangle < \mu + \|v\|_1 \right\}$$





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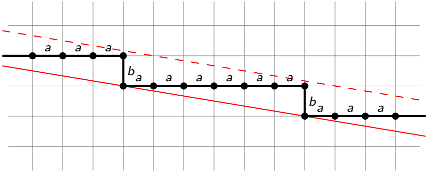
$\mathcal{P}((1, 6), 0)$

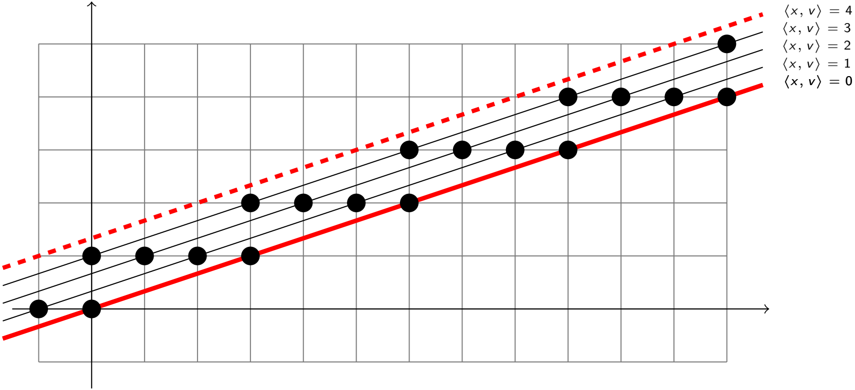
$0 \leq 1x + 6y < 7$



A digital line can be coded on two letters.

Periodic structure of a digital line

- $\langle x, v\rangle$ is the **height** of x,
- $v = (-3, 1)$,
- $\mathcal{P}(v, 0) = \{x \in \mathbb{Z}^2 \mid 0 \leq \langle x, v\rangle < 4\}$.



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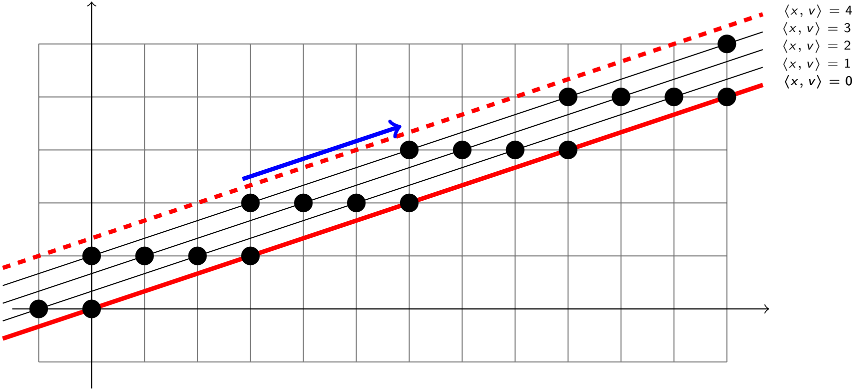
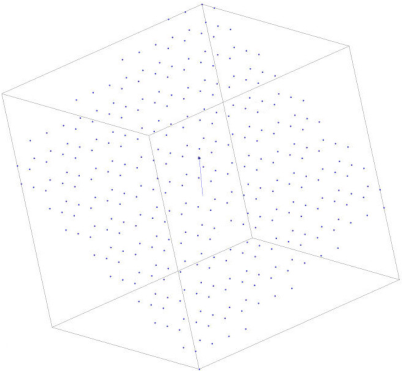
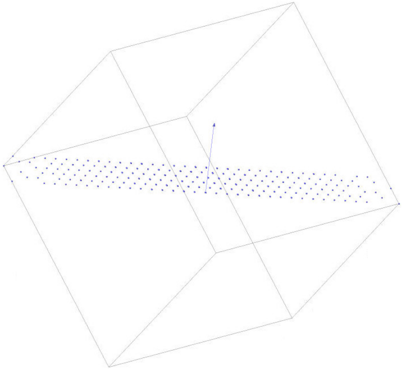


- $\langle x, v\rangle = \langle y, v\rangle \implies y - x$ is a period vector.
- A point of each height from 0 to $\|v\|_1 - 1$ appear in a period.

Periodic structure of a digital plane

$$v = (1,2,3), \quad \mathcal{P}(v,0) = \{x \in \mathbb{Z}^3 \mid 0 \leq \langle x, v \rangle < 6\}$$





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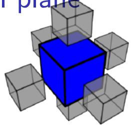

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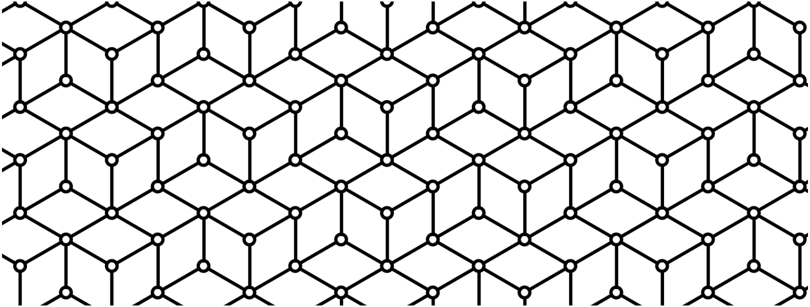

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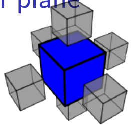

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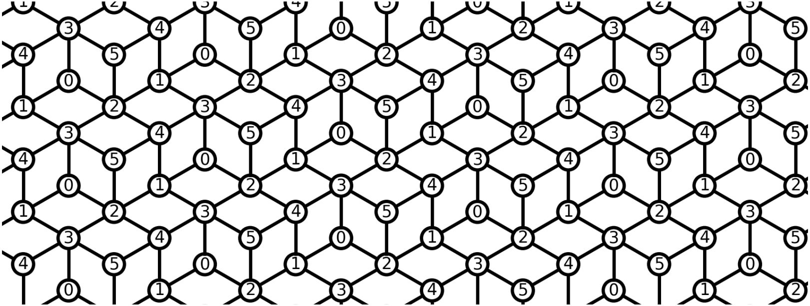

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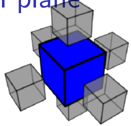

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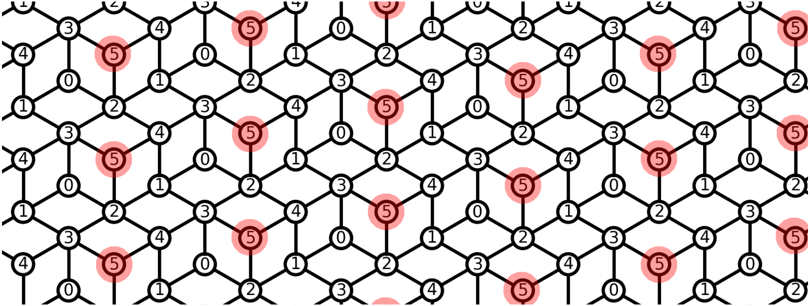

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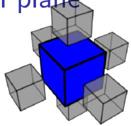

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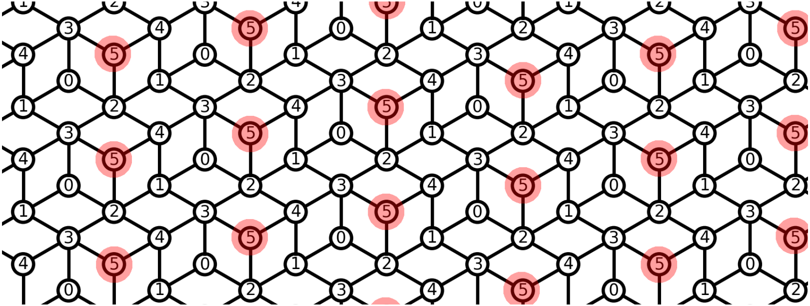

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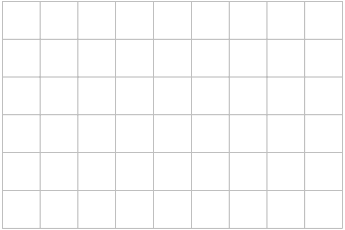
Christoffel words
Digital convexity test

Christoffel words

Definition ([Christoffel 1875])

A **Christoffel word** codes digital path right below a segments between two consecutive integer points



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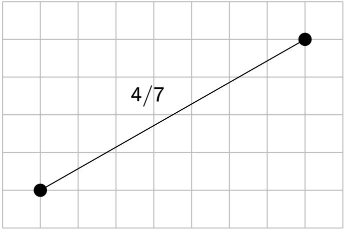

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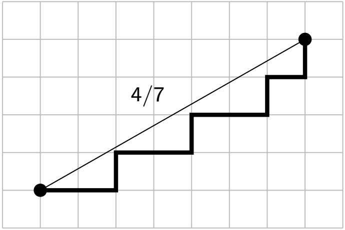

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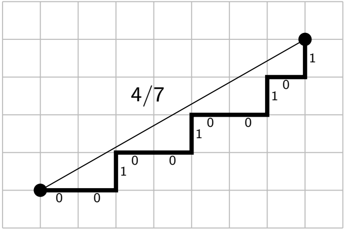

$w = 00100100101$ is the Christoffel word of **slope** $4/7$.

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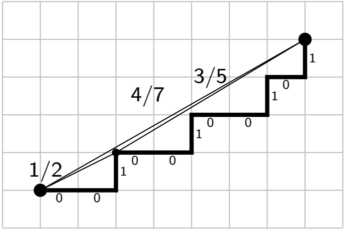


$w = 001 \cdot 00100101$ is the Christoffel word of **slope** $4/7$.

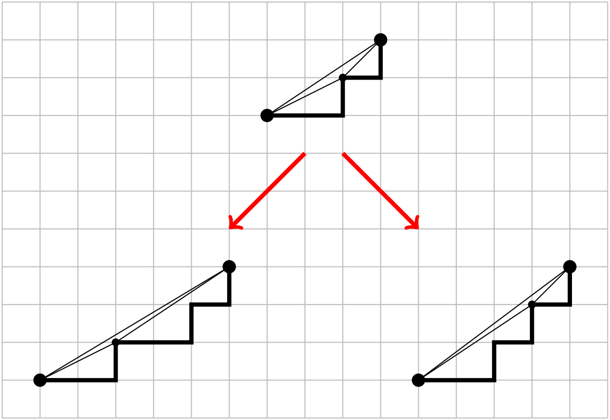
Theorem ([Borel, Laubie 93])

Any Christoffel word, other than 0 *and* 1*, can be written in a unique way as a product of two Christoffel words.*

This is called the **standard factorization**, noted $w = (u, v)$.

Christoffel Tree

If (u, v) is a standard factorization, then (u, uv) and (uv, v) are standard factorizations of Christoffel words.



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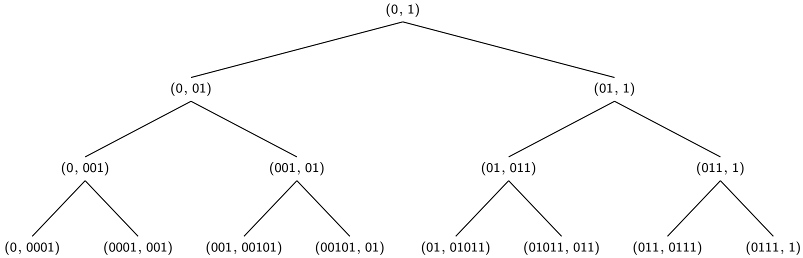
The **Christoffel Tree** is the tree obtained, starting from $(0, 1)$, using the rule :

$$(u, v)$$

$$(u, uv) \qquad\qquad (uv, v)$$



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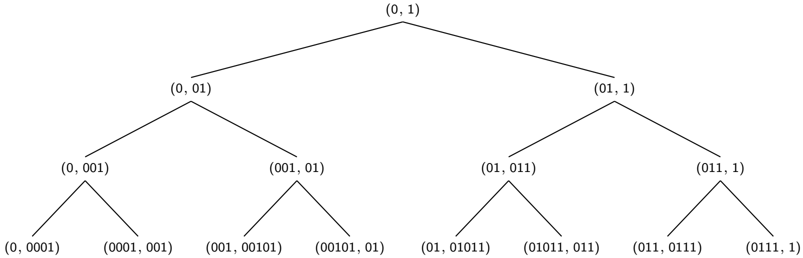
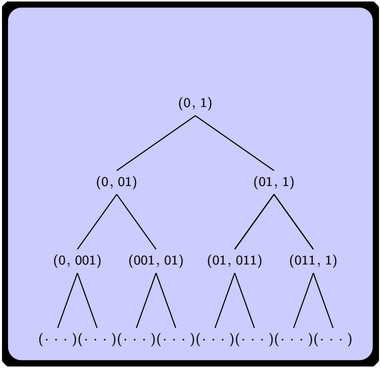
Theorem
Every Christoffel word appears exactly once in the Christoffel Tree.

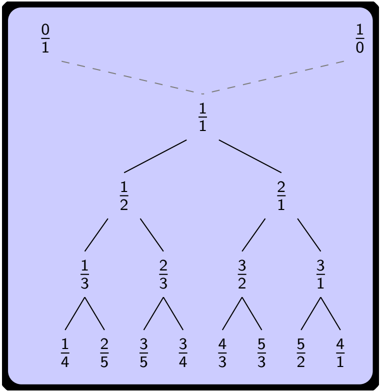


Stern-Brocot Tree

Christoffel tree



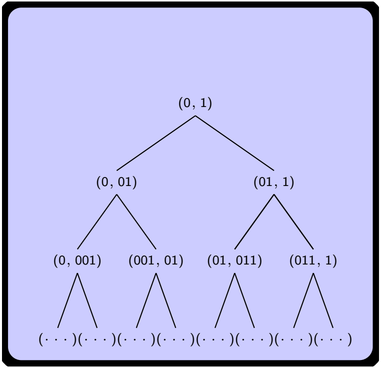
Stern-Brocot tree.



Every irreducible fraction appears exactly once in the Stern-Brocot tree.

Stern-Brocot Tree

Christoffel tree



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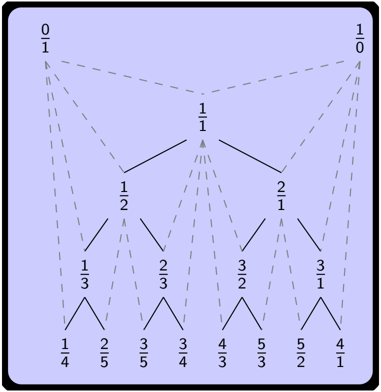

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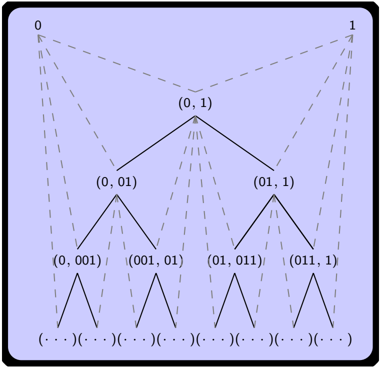

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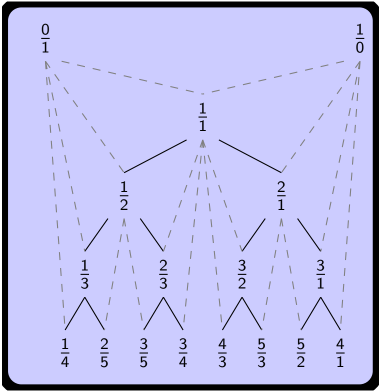

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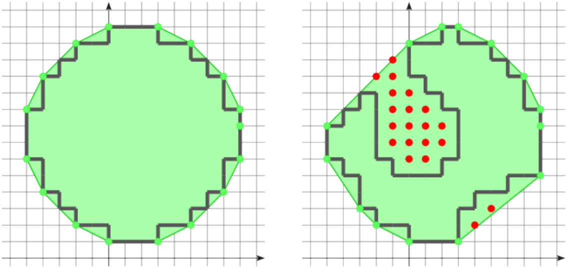
Digital convexity

Definition

A digital set $D \subset Z^d$ is digitally convex if

- $\mathrm{Dig}(\mathrm{Conv}(D)) = D$.



Definitions and characterizations :

- [Minsky and Papert 1969]
- [Sklansky 1970]
- [Kim, Rosenfeld 1981]
- [Hübler, Klette, Voss 1981]
- [Chassery 1983]
- ...
- [Brlek, Lachaud, P., Reutenauer 2009]

Nested prefixes

Corollary

A Christoffel word that admits $w = (u, v)$ as a proper prefix, has a prefix of the form : $w^k v = (w, w^{k-1} v)$.

Identifying the longest prefix that is a Christoffel word :

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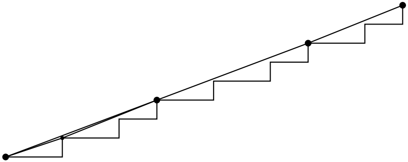

w'			
w		w	v
u	v		

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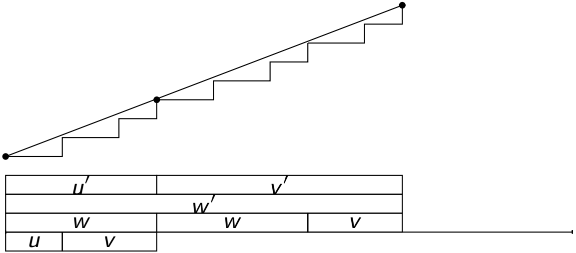

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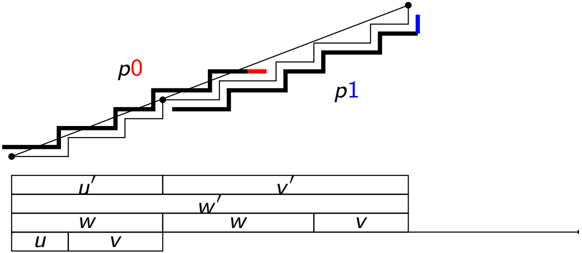

Corollary

Let word $w = (u, v)$ and $v = p1$, then $p0$ is a prefix of w.

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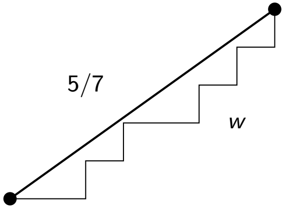
Lexicographic order

Property

Lexicographic order on Christoffel words correspond to the order on the slope



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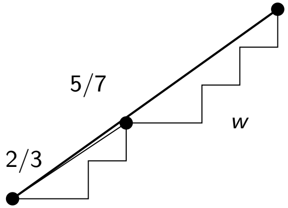

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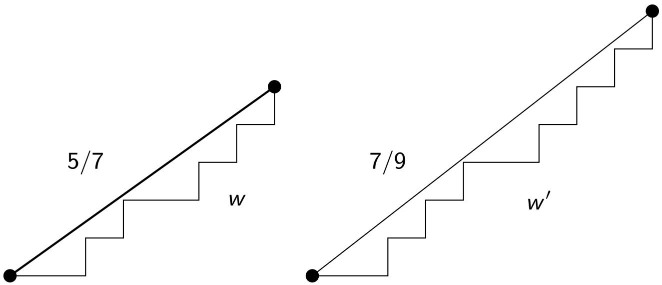

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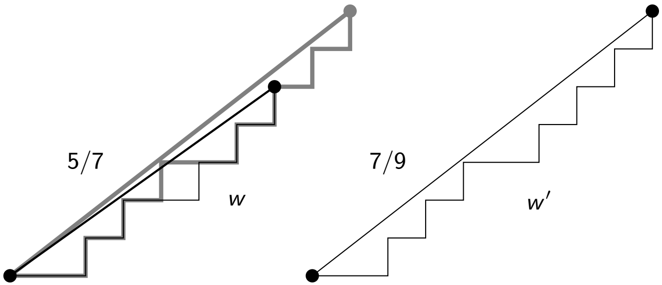

Lyndon words

Definition ([Lyndon 54])

A w is a *Lyndon word* iff for every proper suffix s of w,

$$w <_{\mathbf{Lex}} s$$

Examples :

1. $aabab$ is Lyndon since $aabab <_{\mathbf{Lex}} \{abab, bab, ab, b\}$,
2. $abaab$ is not Lyndon, since $aab <_{\mathbf{Lex}} abaab$.
3. $aabaab$ is not Lyndon, since $aab <_{\mathbf{Lex}} aabaab$.

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Theorem ([Chen, Fox, Lyndon 58])

Every word has a unique factorization as non-increasing Lyndon words

Example :

$$\begin{aligned} & 1\,1\,0\,1\,1\,0\,1\,1\,0\,0\,1\,0\,0\,1\,1\,0\,0\,0 \\ = \; & 1 \cdot 1 \cdot 0\,1\,1 \cdot 0\,1\,1 \cdot 0\,0\,1\,0\,0\,1\,1 \cdot 0 \cdot 0 \cdot 0 \\ = \; & (1)^2 \cdot (0\,1\,1)^2 \cdot (0\,0\,1\,0\,0\,1\,1)^1 \cdot (0)^3. \end{aligned}$$

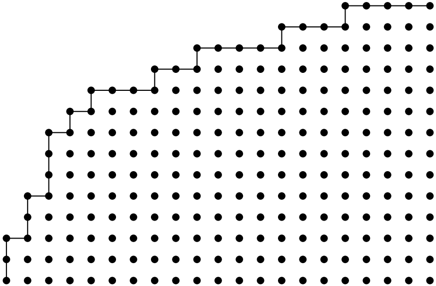
Combinatorial view of convexity

Theorem ([Brlek, Lachaud, P., Reutenauer 09])

The north-west part of a digital shape is convex iff its Lyndon factorization contains only Christoffel words.

Sketch of the proof :

- Uniqueness of the Lyndon factorization.
- No integer points between a Christoffel word and its convex hull.



$$1101101110101000100100001000010000$$

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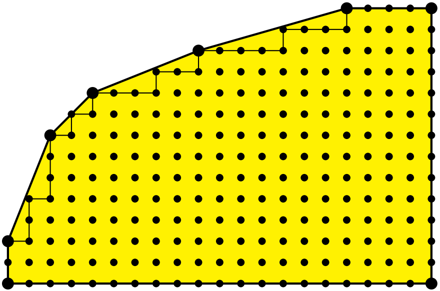

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Duval algorithm

Recursive computation of the First Lyndon Prefix (FLF),

$w =$

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$$w = \underbrace{\boxed{l_0}\boxed{l_0}\boxed{l_0}}\_\_\_\_\_\_\_\_$$

1. Let l_0 be a Lyndon prefix and k be it's number of repetitions.

Duval algorithm

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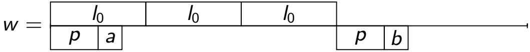

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2. Identify at the first letter that is not that same than in l_0.

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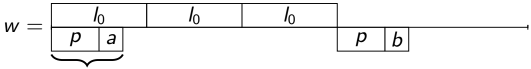

If $a > b$ then the Lyndon fact. starts by l_0^k

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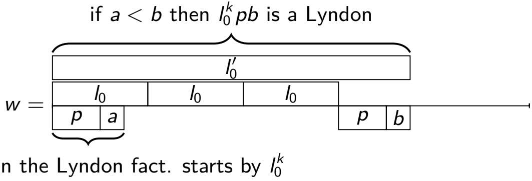
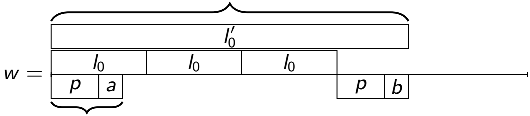


1. Let l_0 be a Lyndon prefix and k be it's number of repetitions.
2. Identify at the first letter that is not that same than in l_0.
3. If its smaller than l_0 is FLF, otherwise, $l_0^k pb$ is a Lyndon word.

Duval algorithm

Recursive computation of the First Lyndon Prefix (FLF),

if $a < b$ then $l_0^k pb$ is a Lyndon



If $a > b$ then the Lyndon fact. starts by l_0^k

$$\underbrace{0}_{l_0}\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1$$

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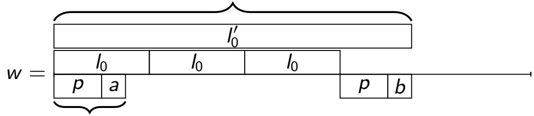
3. If its smaller than l_0 is FLF, otherwise, $l_0^k pb$ is a Lyndon word.

Duval algorithm

Recursive computation of the First Lyndon Prefix (FLF),

if $a < b$ then $l_0^k pb$ is a Lyndon



If $a > b$ then the Lyndon fact. starts by l_0^k

$\downarrow$ $\downarrow$

0 0 1 0 0 1 0 1 0 0 1 0 0 1 0 0 0 1 0 0 1

l_0

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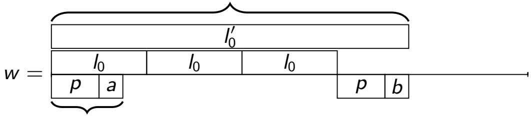

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$$\begin{array}{cccccccccccccccccccccccc} \downarrow & & \downarrow \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ l_0 & l_0 \end{array}$$

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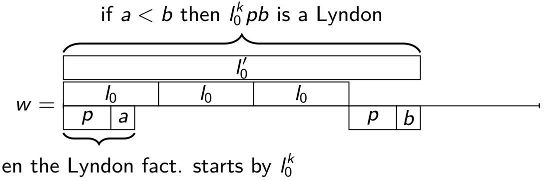

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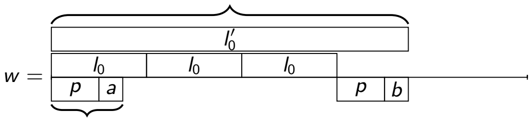

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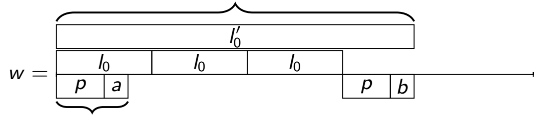

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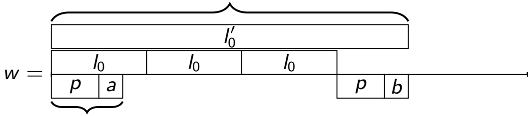

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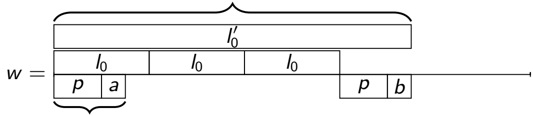

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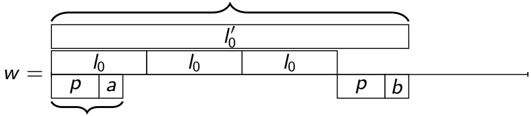

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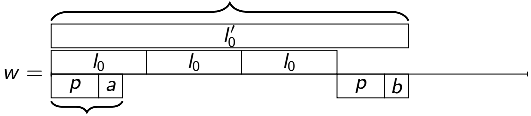

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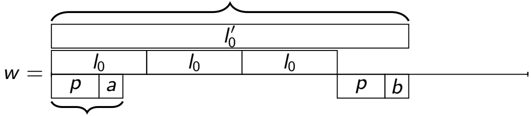

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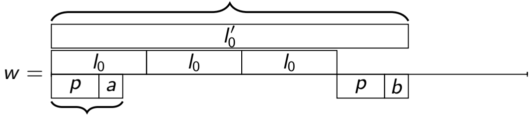

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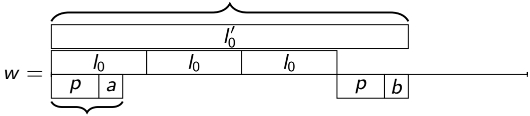

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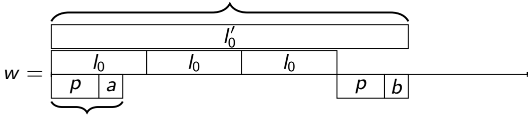

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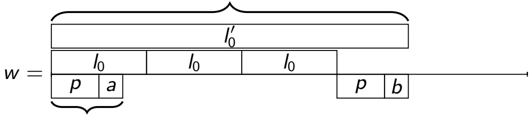

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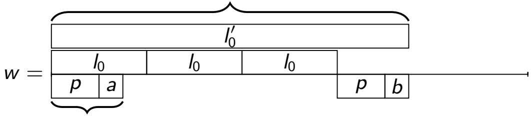

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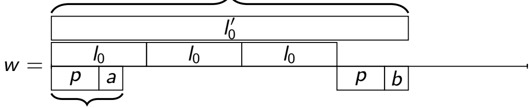

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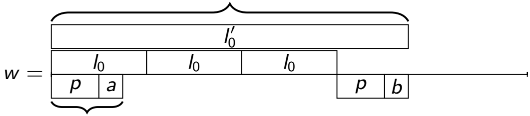

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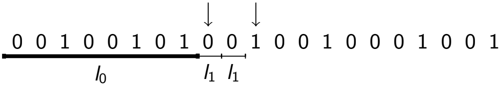

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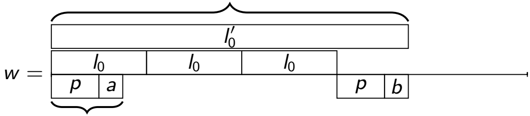

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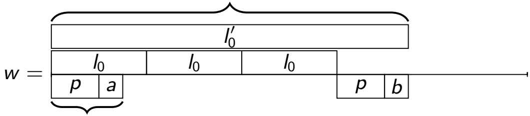

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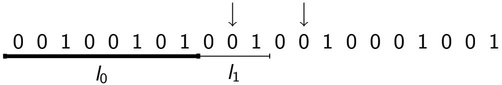

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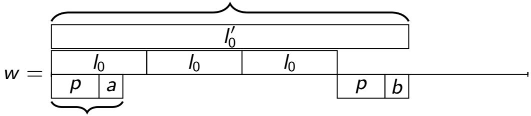

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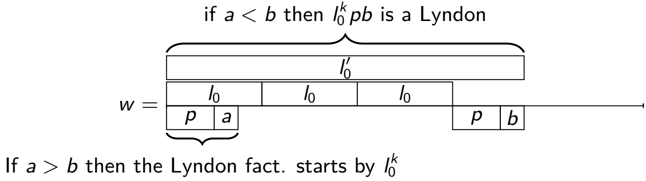

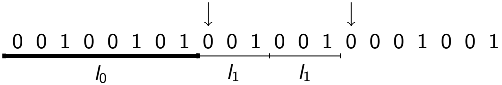

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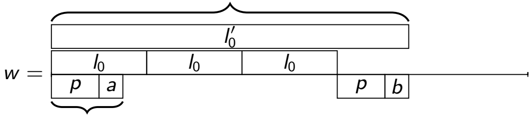

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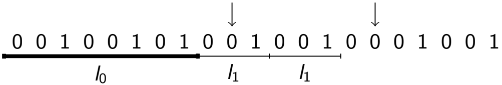

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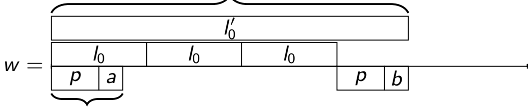

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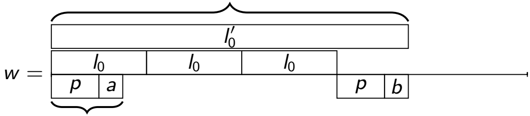

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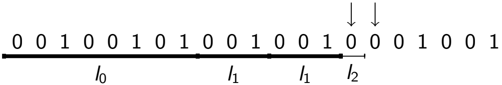

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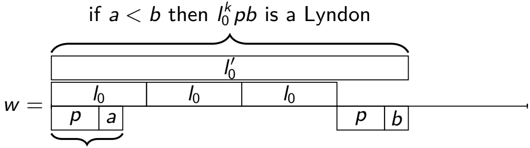

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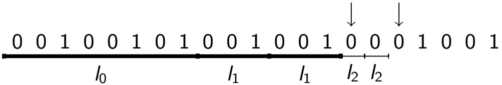

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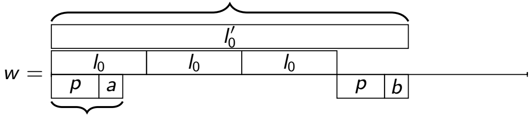

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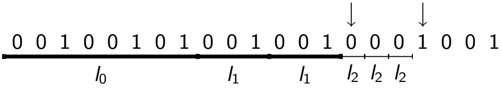

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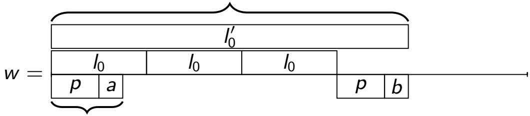
2. Identify at the first letter that is not that same than in l_0.
3. If its smaller than l_0 is FLF, otherwise, $l_0^k pb$ is a Lyndon word.

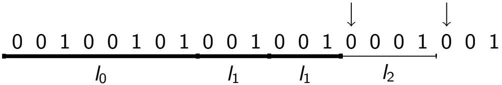
Duval algorithm

Recursive computation of the First Lyndon Prefix (FLF),

if $a < b$ then $l_0^k pb$ is a Lyndon



If $a > b$ then the Lyndon fact. starts by l_0^k



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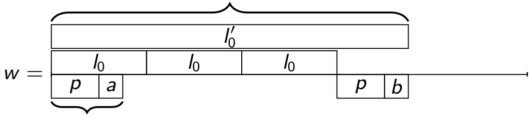

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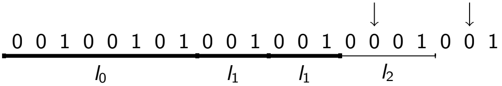

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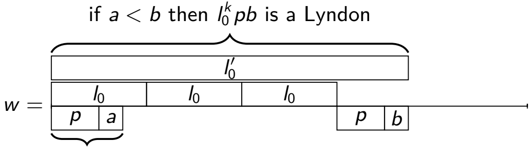

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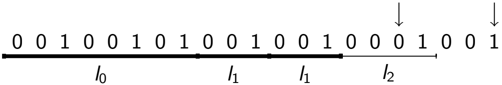

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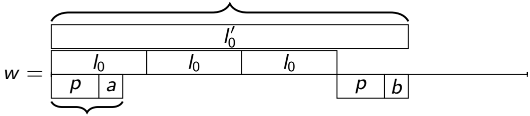

If $a > b$ then the Lyndon fact. starts by l_0^k

$$\underbrace{0\,0\,1\,0\,0\,1\,0\,1}_{l_0}\;\underbrace{0\,0\,1}_{l_1}\;\underbrace{0\,0\,1}_{l_1}\;\underbrace{0\,0\,0\,1\,0\,0\,1}_{l_2}$$

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Duval algorithm

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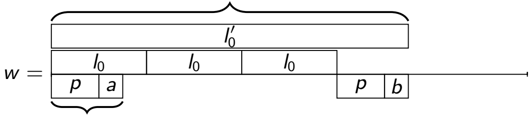

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$$\underbrace{0\ 0\ 1\ 0\ 0\ 1\ 0\ 1}_{l_0}\ \underbrace{0\ 0\ 1}_{l_1}\ \underbrace{0\ 0\ 1}_{l_1}\ \underbrace{0\ 0\ 0\ 1\ 0\ 0\ 1}_{l_2}$$

1. Let l_0 be a Lyndon prefix and k be it's number of repetitions.
2. Identify at the first letter that is not that same than in l_0.
3. If its smaller than l_0 is FLF, otherwise, $l_0^k pb$ is a Lyndon word.

   When comparing two different letters, let $l_0 = (u, v)$:
   - if $|pb| = |v|$ then
     $a = 0$ and $b = 1$ and l_0' is a Christoffel word.
   - if $|pb| \neq |v|$ and $a = 1$ and $b = 0$ then
     l_0 is the first edge of the convex hull.
   - if $|pb| \neq |v|$ and $a = 0$ and $b = 1$ then
     Shape is not convex.

Part II

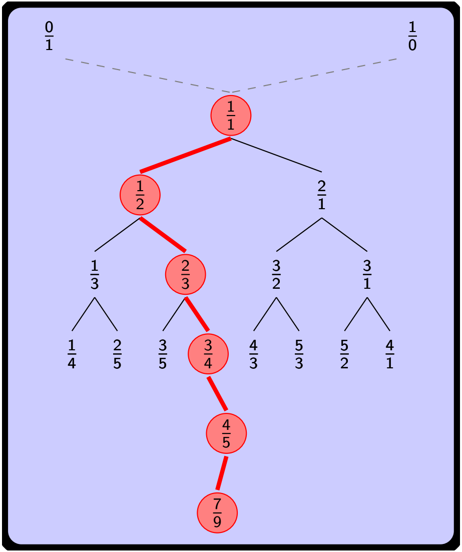
Construction guided by Euclid

5 From Euclid to Christoffel

6 Alternative construction

Euclid Algorithm

Stern-Brocot tree



Euclid algorithm	Approximation
$(7, 9)$	$(1, 1)$
$\downarrow$	$\downarrow$
$(7, 2)$	$(1, 2)$
$\downarrow$	$\downarrow$
$(5, 2)$	$(2, 3)$
$\downarrow$	$\downarrow$
$(3, 2)$	$(3, 4)$
$\downarrow$	$\downarrow$
$(1, 2)$	$(4, 5)$
$\downarrow$	$\downarrow$
$(1, 1)$	$(7, 9)$

Matricial view

	Euclid algorithm	Approx.
n	v_n	a_n
0	$(\underline{7}, 9)$	$(1, 1)$
	$\downarrow$	$\downarrow$
1	$(7, \underline{2})$	$(1, 2)$
	$\downarrow$	$\downarrow$
2	$(5, \underline{2})$	$(2, 3)$
	$\downarrow$	$\downarrow$
3	$(3, \underline{2})$	$(3, 4)$
	$\downarrow$	$\downarrow$
4	$(\underline{1}, 2)$	$(4, 5)$
	$\downarrow$	$\downarrow$
5	$(1, 1)$	$(7, 9)$

Euclid algorithm
Given a vector (x, y), return

- $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ if $x < y$,
- $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ if $x > y$,
- **stop** if $x = y$.

Given a vector $v \in (\mathbb{N} \setminus \{0\})^2$, let :

- $v_0 = v$,
- For all $n \geq 1$: $\begin{cases} M_n = \textbf{Euclid}(v_{n-1}) \\ v_n = M_n v_{n-1}. \end{cases}$

Matricial view

	Euclid algorithm	Approx.
n	v_n	a_n
0	$(\underline{7}, 9)$	$(1,1)$
	$\downarrow$	$\downarrow$
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	$\downarrow$	$\downarrow$
2	$(5, \underline{2})$	$(2,3)$
	$\downarrow$	$\downarrow$
3	$(3, \underline{2})$	$(3,4)$
	$\downarrow$	$\downarrow$
4	$(\underline{1}, 2)$	$(4,5)$
	$\downarrow$	$\downarrow$
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Euclid algorithm
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- $v_0 = v$,
- For all $n \geq 1$: $\begin{cases} M_n = \textbf{Euclid}(v_{n-1}) \\ v_n = M_n v_{n-1}. \end{cases}$

Property

- $v_n = M_n M_{n-1} \cdots M_1 v$
- $a_n = M_1^{-1} M_2^{-1} \cdots M_n^{-1} \binom{1}{1}$

Matricial view

Lemma

Let A, B, C be Christoffel words such that $C = (A, B)$ and $\overrightarrow{C} = a_n$. Let $\overrightarrow{A} = (A_x, A_y)$, $\overrightarrow{B} = (B_x, B_y)$, then:

$$M_1^\top M_2^\top \cdots M_n^\top = \begin{bmatrix} A_x & -B_x \\ -A_y & B_y \end{bmatrix}$$

Proof. By recurrence. True for $n = 0$, $\mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Suppose true for n,

(A, B)

(A, AB) (AB, B)

$$M_1^\top \cdots M_{n+1}^\top = \begin{bmatrix} A_x & -B_x \\ -A_y & B_y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_x & -A_x - B_x \\ -A_y & A_y + B_y \end{bmatrix}.$$

Matricial view

Lemma

Let A, B, C be Christoffel words such that $C = (A, B)$ and $\overrightarrow{C} = a_n$.
Let $\overrightarrow{A} = (A_x, A_y)$, $\overrightarrow{B} = (B_x, B_y)$, then:

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$$\begin{array}{ccc} & (A, B) & \\ \swarrow & & \searrow \\ (A, AB) & & (AB, B) \end{array}$$

$$M_1^\top \cdots M_{n+1}^\top = \begin{bmatrix} A_x & -B_x \\ -A_y & B_y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_x & -A_x - B_x \\ -A_y & A_y + B_y \end{bmatrix}.$$

$$M_1^\top \cdots M_n^\top e_1 = (A_x, -A_y)$$

$$M_1^\top \cdots M_n^\top e_2 = (-B_x, B_y)$$

The Translation-Union Construction

Construction
[Domenjoud, Vuillon 12],
[Berthé, Jamet, Jolivet, P. 2013]

Let $v_0 = v$, $B_0 = \{\mathbf{0}\}$ and for all $n \geq 1$ let :

M_n : the matrix selected from v_{n-1},

$$v_n = M_n v_{n-1}$$

δ_n : the index of the coordinate of v_{n-1} that is subtracted,

$$T_n = M_1^\top \cdots M_n^\top e_{\delta_n}, \qquad \textit{(translation)}$$

$$B_n = B_{n-1} \cup (T_n + B_{n-1}), \qquad \textit{(body)}$$

$$H_n = \sum_{i \in \{1,\ldots,n\}} T_i, \qquad \textit{(highest point)}$$

$$L_n = H_n + \{M_1^\top \cdots M_n^\top e_i\}. \qquad \textit{(legs)}$$

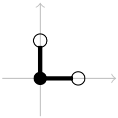
Note that:

$H_n \in B_n$,

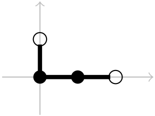
$L_n \cap B_n = \emptyset$.

$\bullet \in B_n$, $\circ \in L_n$

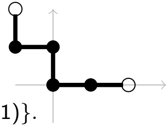
$v_0 = (2,3)$,
$a_0 = (1,1)$
$H_0 = (0,0)$,
$L_0 = \{(1,0),(0,1)\}$.



$v_1 = (2,1), \delta_1 = 1$
$a_1 = (1,2)$
$T_1 = (1,0)$
$H_1 = (1,0)$,
$L_1 = \{(2,0),(0,1)\}$.



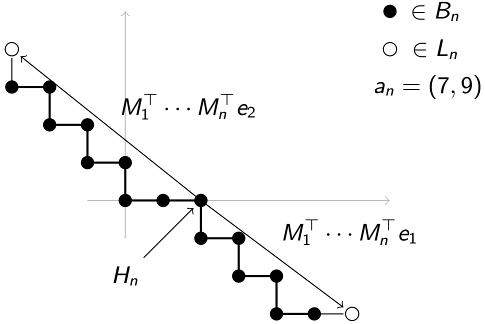
$v_2 = (1,1), \delta_2 = 2$
$a_2 = (2,3)$
$T_2 = (-1,1)$
$H_2 = (0,1)$,
$L_1 = \{(2,-1),(-1,1)\}$.



The Translation-Union Construction

Property

The points of $B_n \cup L_n$ for the Christoffel word of vector a_n. Moreover, let $\{x, y\} = L_n$ then $\langle x, a_n\rangle = \langle y, a_n\rangle$.



Part III

Generalization to higher dimensions

7 A general construction

8 The fully subtractive algorithm

3D continued fraction algorithms

Euclid algorithm : given two number subtract the smaller to the larger.
$(7,9) \to (7,2) \to (5,2) \to (3,2) \to (1,2) \to (1,1) \to (1,0)$

3D continued fraction algorithms

Euclid algorithm : given two number subtract the smaller to the larger.
$(7,9) \to (7,2) \to (5,2) \to (3,2) \to (1,2) \to (1,1) \to (1,0)$

Given three numbers :

- **Selmer** : subtract the smallest to the largest.
  $(3,7,5) \to (3,4,5) \to (3,4,2) \to (3,2,2) \to (1,2,2) \to (1,2,0) \to (1,1,0) \to (1,0,0)$.
- **Brun** : subtract the second largest to the largest.
  $(3,7,5) \to (3,2,5) \to (3,2,2) \to (1,2,2) \to (1,2,0) \to (1,1,0) \to (1,0,0)$.
- **Fully subtractive** : subtract the smallest to the two others.
  $(3,7,5) \to (3,4,2) \to (1,2,2) \to (1,1,1) \to (1,0,0)$.
- **Poincaré** : subtract the smallest to the mid and the mid to the largest.
  $(3,7,5) \to (3,2,2) \to (1,2,0) \to (1,1,0) \to (1,0,0)$.
- **Arnoux-Rauzy** : subtract the sum of the two smallest to the largest (not always possible).
  $(3,7,5) \to \text{impossible}$.
- ...

The Translation-Union Construction

Construction

Let $v_0 = v$, $B_0 = \{\mathbf{0}\}$ and for all $n \geq 1$ let :

M_n : the matrix selected from v_{n-1},

$$v_n = M_n v_{n-1}$$

δ_n : the index of the coordinate of v_{n-1} that is subtracted,

$$T_n = M_1^\top \cdots M_n^\top e_{\delta_n}, \qquad (translation)$$

$$B_n = B_{n-1} \cup (T_n + B_{n-1}), \qquad (body)$$

$$H_n = \sum_{i\in\{1,\ldots,n\}} T_i, \qquad (highest\ point)$$

$$L_n = H_n + \{M_1^\top \cdots M_n^\top e_i\}. \qquad (legs)$$

The Translation-Union Construction

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Let $v_0 = v$, $B_0 = \{\mathbf{0}\}$ and for all $n \geq 1$ let :

M_n : the matrix selected from v_{n-1},

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$$H_n = \sum_{i\in\{1,\dots,n\}} T_i, \qquad (\textit{highest point})$$

$$L_n = H_n + \{M_1^\top \cdots M_n^\top e_i\}. \qquad (\textit{legs})$$

Property

If the action of M_n is to subtract a coordinate to at least one other coordinate while keeping it positive, then $B_n \in \mathcal{P}(v, 0)$.

Proof : $\langle T_n, v\rangle = \langle M_1^\top \dots M_n^\top e_{\delta_n}, v\rangle = \langle e_{\delta_n}, M_n \cdots M_1 v\rangle = \langle e_{\delta_n}, v_n\rangle$ is equal to the value of the coordinate that is subtracted.

Let $x \in B_n$, then $x = \sum_{i\in I} T_i$ for some $I \subset \{1, \cdots, n\}$ and

$$0 \leq \langle x, v\rangle < \|v\|_1$$

Construction using fully Subtractive

The **fully subtractive** algorithm :

Subtract the smallest coordinate to the two others.

The matrices are :

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Construction using fully Subtractive

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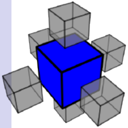
Definition

Let $\mathcal{K}$ be the set of vectors v such $\mathbf{FS}^N(v) = (1,1,1)$ for some $N \geq 1$.

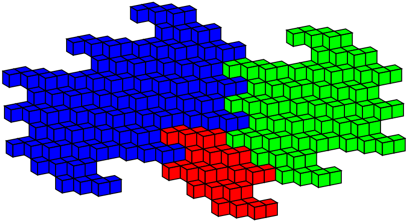
- $\mathcal{K} \ni (1,2,2) \xrightarrow{\mathbf{FS}} (1,1,1)$
- $\mathcal{K} \not\ni (2,2,5) \xrightarrow{\mathbf{FS}} (0,2,3)$

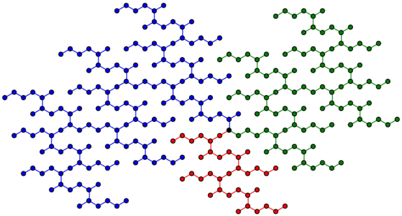
Tree structure

Theorem ([Domenjoud, Vuillon 12])
When using the fully subtractive algorithm, the graph of B_n is a tree.

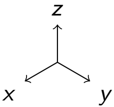


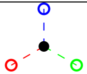
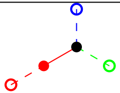
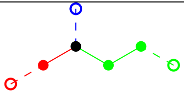
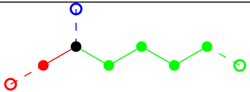
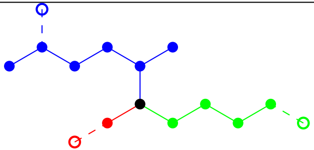
Example : $v = (136, 184, 249) \in \mathcal{K}$





Recursive construction with Fully Subtractive



$B_n \cup L_n$	Approx.	Fully subtractive algorithm
	$(1,1,1)$	$(6,8,11)$
	$(1,2,2)$	$(6,2,5)$
	$(2,3,4)$	$(4,2,3)$
	$(3,4,6)$	$(2,2,1)$
	$(6,8,11)$	$(1,1,1)$

Recursive construction with Fully Subtractive

Property

Using Fully Subtractive on $v \in \mathcal{K}$, let N be such that $v_N = (1,1,1)$ and so $a_N = v$:

1. $B_N \cup L_N$ *is connected.*
2. B_N *has exactly one point at each height from* 0 *to* $\left\lfloor \frac{\|v\|_1}{2} \right\rfloor - 1$
3. *All points of* L_N *have height* $\left\lfloor \frac{\|v\|_1}{2} \right\rfloor$

Recursive construction with Fully Subtractive

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1. B_n is a tree.

Recursive construction with Fully Subtractive

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1. B_n is a tree.
2. $v = v_0 \xrightarrow{\textbf{FS}} v_1 \xrightarrow{\textbf{FS}} \cdots \xrightarrow{\textbf{FS}} v_N = (1,1,1)$

   The heigth of each T_i is equal to the coordinate that has been subtracted to the two other coordinates.

   $\|v_n\|_1 = \|v_{n-1}\|_1 - 2\langle T_n, v\rangle$.

Recursive construction with Fully Subtractive

Property

Using Fully Subtractive on $v \in \mathcal{K}$, let N be such that $v_N = (1,1,1)$ and so $a_N = v$:

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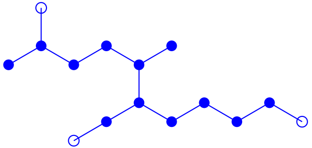
   The heigth of each T_i is equal to the coordinate that has been subtracted to the two other coordinates.

   $\|v_n\|_1 = \|v_{n-1}\|_1 - 2\langle T_n, v\rangle.$
3. $L_n = H_n + \{M_1^T \cdots M_n^T e_i\}$ and
   $\langle M_1^T \cdots M_N^T e_i, v\rangle = \langle e_i, M_N \cdots M_1 v\rangle = \langle e_i, v_N\rangle = \langle e_i, (1,1,1)\rangle = 1.$

From pattern to digital plane

$v = (6, 8, 11), \left\lfloor \frac{\|v\|_1}{2} \right\rfloor = 12$



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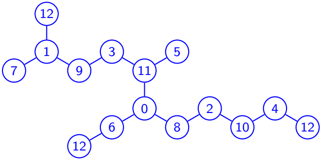

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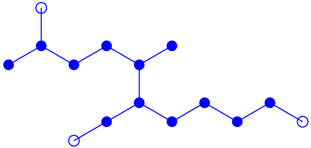

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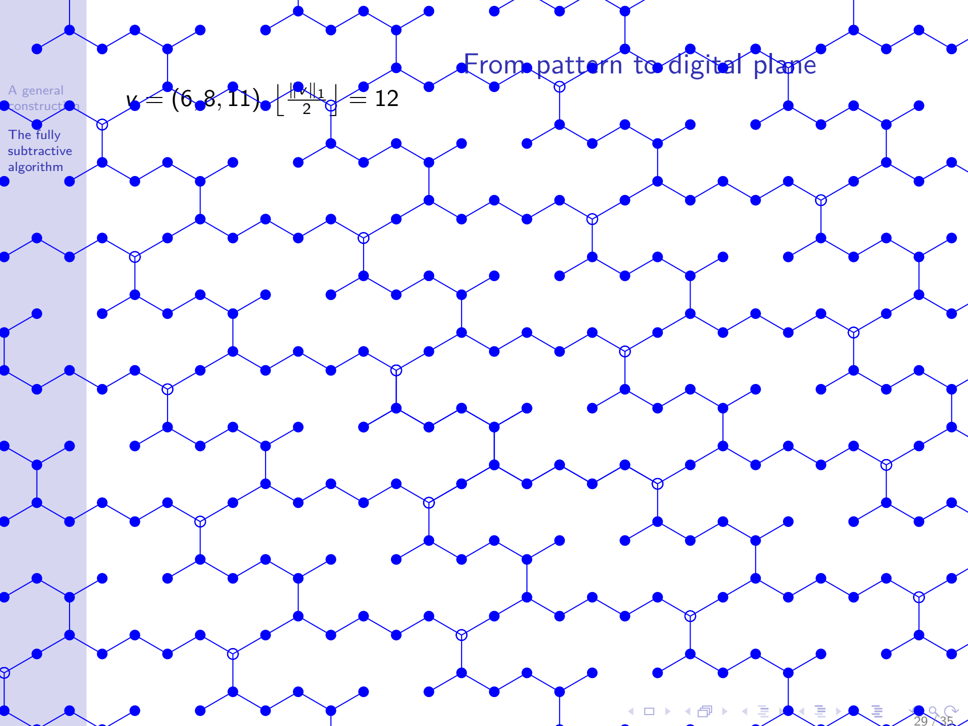

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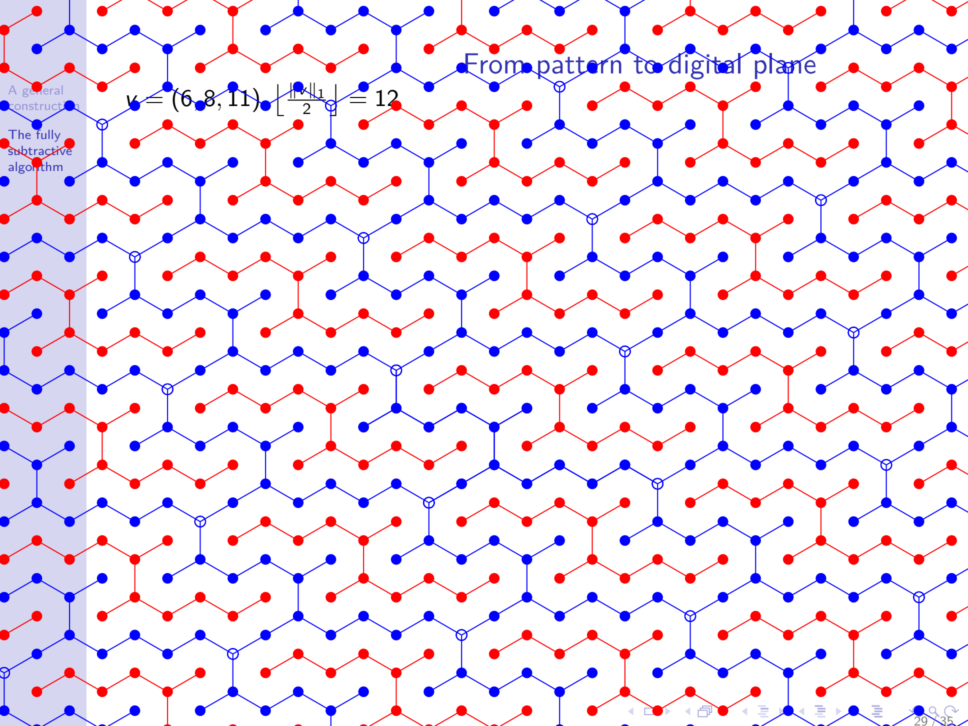

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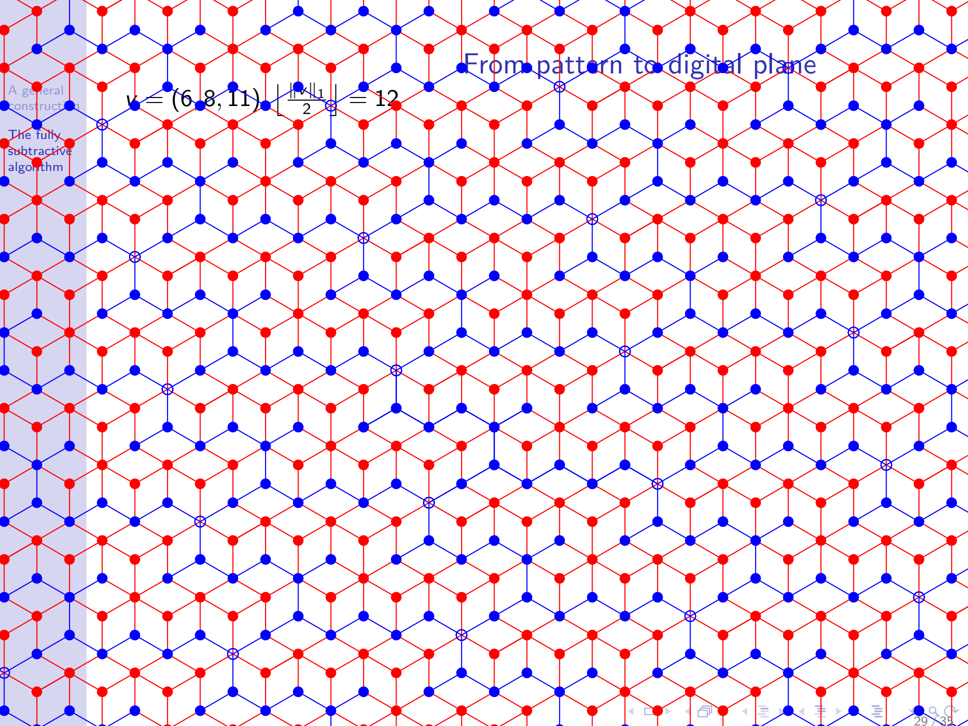
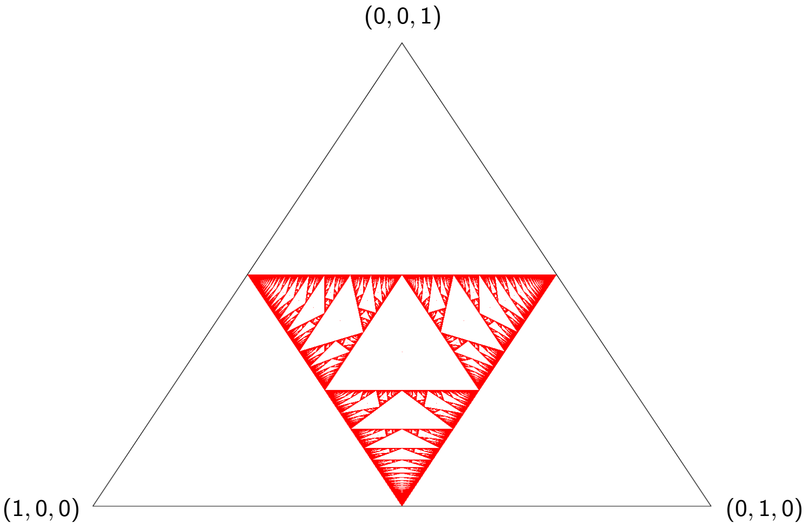


The set $\mathcal{K}$

$$v \xrightarrow{\mathsf{FS}} \cdots \xrightarrow{\mathsf{FS}} (1,1,1)$$



Vectors not in $\mathcal{K}$

Let $v \in (\mathbb{N} \setminus \{0\})^3$ such that $v \notin \mathcal{K}$, then either :

1. $\mathbf{FS}^n(v) = (g, g, g)$ with $g \geq 2$.
2. $\mathbf{FS}^n(v) = (a, a, b)$ with $a < b$ so that $\mathbf{FS}((a, a, b)) = (0, a, b - a)$.
3. $\mathbf{FS}^n(v) = (a, b, c)$ with $a + b \leq c$.

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Solution:

1. Then $g = \gcd(v)$, use $v/g \in \mathcal{K}$.

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Solution:

1. Then $g = \gcd(v)$, use $v/g \in \mathcal{K}$.
2. Do not use **FS**...
3. Do not use **FS**...

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Solution:

1. Then $g = \gcd(v)$, use $v/g \in \mathcal{K}$.
2. Do not use **FS**. . .
3. Do not use **FS**. . .

. . . ok but *what else* ?

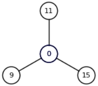
Vectors not in $\mathcal{K}$

Idea : Use hybrid algorithm, suppose $a \leq b \leq c$,

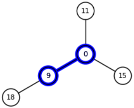
$$(a,b,c) = \begin{cases} \mathbf{FS}((a,b,c)) \text{ if } a \neq b \text{ and } a+b \leq c, \\ \mathbf{Brun}((a,b,c)) \text{ otherwise.} \end{cases}$$

Brun: subtract the second biggest coordinate to the biggest one.

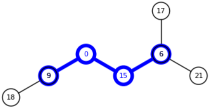
$(9, 15, 11)$



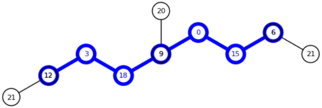
$\xrightarrow{\mathbf{FS}} (9, 6, 2)$



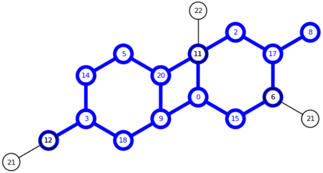
$\xrightarrow{\mathbf{Brun}} (3, 6, 2)$



$\xrightarrow{\mathbf{Brun}} (3, 3, 2)$



$\xrightarrow{\mathbf{FS}} (1, 1, 2)$



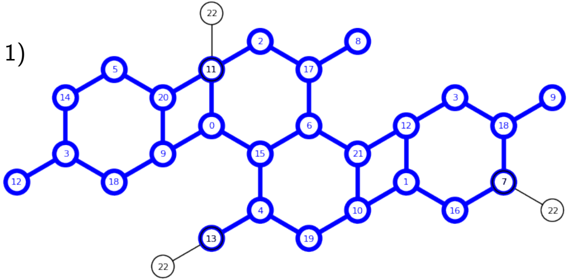
The hybrid **FS+Brun** algorithm

Property ([Lafrenière, Jamet, P.])

*Using the hybrid **FS+Brun** algorithm, for all vector $v \in (\mathbb{N} \setminus \{0\})^3$*

1. $\exists N$ *such that* $v_N = (1,1,1)$ *(or* gcd. . . *).*
2. *Vectors of* L_n *have same height, (providing period vectors).*
3. $B_n \cup L_n$ *is connected but in general not a tree.*
4. $\left\lfloor \frac{\|v\|_1}{2} \right\rfloor - 1 \leq \langle H_N \rangle < \|v\|_1.$
5. *There is a least one point at each height from* 0 *to* $\langle H_N \rangle$ *but in general no unicity.*

$\xrightarrow{\textbf{Brun}} (1,1,1)$



Conclusion

Good:

- Generalization of Christoffel words to higher dimensions.
- Construction is recursive and based on continued fraction algorithms.
- Construction of the periodic pattern of the digital plane for $\mathcal{K}$.

~~Problems:~~ Open questions :

- Provide a gcd algorithm that builds minimal patterns for $\mathcal{K}^{\mathsf{C}}$.
- Give a geometrical interpretation of the patterns produced by the hybrid algorihtm.
- Control the anisotropy of the patterns (avoid stretched forms in favor of *potato-likeness*).
- Apply recursive structure to image analysis algorithms.

Merci