Metric graph model of fibrous material

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Fibrous materials

Appear in many fields
- Galaxy superclusters
- Reinforced concrete
- Internal bone structure
- Catalytic foam
Fibrous materials

Appear in many fields
- Galaxy superclusters
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- Gas/gas or gas/solid reactions
- Various materials
  - Metal (Cu, Al)
  - Plastic (Polyurethane)
- Large stiffness to mass ratio
- High surface to volume ratio

Thermal bench (LGPC)
see posters P-70 and P-50
Catalytic foam

- Non repeatable manufacturing process
- Internal inhomogeneities
- Complex non planar structures
- Multiple scales
Metric graph model

- Fibrous material properties
  - Topology: cells, holes and cycles
  - Geometry: lengths, areas et volumes

- Metric graph
  - Graph = nodes + edges
  - Metric graph embedding = edge lengths
  - 1D differential manifold
Manifold embedding

- Manifold = smooth curved space with boundaries and holes
- Tensorial linear operators, scalar and vector fields
- Geodesic length = shortest length inside manifold

- Foam operators projected on metric graph
- Linear PDE resolution
  - Heat flow
  - Electrical resistivity
  - Mechanical stress
Proposed toolkit

- Morphological closing
- Homotopic thinning
  - Iterative topological skeleton simplification
  - Distance map driven
  - Topology conservation / size reduction
- Discrete Exterior Calculus (DEC)
  - Modern versatile discretization scheme
  - Classical linear operators $\nabla, \Delta, \nabla \cdot$ and $\nabla \wedge$
  - Stokes theorem holds $\Rightarrow$ unique convergent solution
  - Khalimsky space $\Rightarrow$ trivial well conditioned discretization
- Geodesic in heat [CRANE13]
- Metric graph reconstruction [AANJANEYA12]
Mathematical morphology

- Close hollow tube
- Noise reduction after thresholding

Original  Dilation  Erosion

Closing = Erosion \circ Dilation  Opening = Dilation \circ Erosion
Mathematical morphology

- Close hollow tube
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Original

Dilation

Erosion

Closing = Erosion \circ Dilation

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Homotopic thinning

Size reduction with topology conservation

- Iterative erosion process
- No hole filling or closing
- Non unique solution

Local simplicity test $\Rightarrow$ efficient (LUT)
Distance weighed priority queue $\Rightarrow$ keep skeleton centered
May be geometrically inaccurate
Continuous Riemannian manifold

- Smooth curved space $M$ with boundary $\partial M$
- Points $p, q$
- Coordinates system $x^1, x^2, ...$
- 0-form $\equiv$ scalar field
- 1-form $\equiv$ vector field $\in T_p M$ (local tangent vector space)

Base operators
- $\ast$ Hodge (orthogonality)
- $\wedge$ wedge (span)
- $d$ exterior derivative
Discrete exterior calculus (DEC)

- Cellular complex (with dual)
- 0-cells ↔ point, 1-cells ↔ edge, ...
- k-cells holds discrete k-form values
- Khalimsky space
- Convergent operators

Stokes’ theorem
\[ \int_M d\omega = \int_{\partial M} \omega \]

Classical operators

- Gradient \( \nabla = d \)
- Divergence \( \nabla \cdot = \ast d \ast \)
- Curl \( \nabla \wedge = d \ast \)
- Laplacian \( \Delta = \ast d \ast d \)
Operators chain and cochain

\[ f \text{ scalar field (0-form)} \]
\[ \nabla f = (d_0 f)^\# \]
\[ \Delta f = \nabla \cdot \nabla f = *_{2}' d_1' *_1 d_0 f \]

\[ V \text{ vector field (1-form\#)} \]
\[ \nabla \cdot V = *_{1}' d_1' \]
\[ \nabla \wedge V = (*_{2}' d_1 (V)^b)^\#' \]
Linear differential equation $\Rightarrow$ Linear algebra problem

- Discretization of forms $\Rightarrow$ discretization of linear operators
- $k$-form $\sim$ vector
- Operator $\sim$ matrix
- Composition $\sim$ matrix multiplication
- Induction of properties (SDP, ...)

\[ \star'_{2} \]
\[ d'_{1} \]
\[ \star_{1} \]
\[ d_{0} = \Delta \]
Geodesic in heat \cite{CRANE13}

1. Heat flow
   \[
   \frac{\partial u}{\partial t} = \Delta u
   \]
   
   \[t \ll 1\]
   
   \[u(t = 0) = u_0\]

2. \(\nabla\) normalization
   \[
   X = \frac{\nabla u}{|\nabla u|}
   \]

3. Poisson
   \[
   \Delta \phi = \nabla \cdot X
   \]

4. \(u_0 = \delta \Rightarrow\)
   
   \(\phi\) distance map
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**Geodesic in heat** \[\text{[CRANE13]}\]
Geodesic in heat [CRANE13]

- Fast (prefactored linear problem)
- Precise (absolute error $\sim 0 - 3$ px)
- Triangular inequality may not hold everywhere
Degree estimation at each point of the digital

Degree = # connected components in spherical cap

Spherical caps built using geodesic distance

Constant degree cluster stands for metric graph elements

- degree = 0 ⇔ isolated node
- degree = 1 ⇔ dangling edge node
- degree = 2 ⇔ edge
- degree > 2 ⇔ node

Proof of convergence if pillars are thin enough
Metric graph reconstruction [AANJANEYA12]
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Degree = 1

Degree = 2

Degree = 3
Metric graph reconstruction [AANJANEYA12]
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Principal component analysis (PCA) on metric graphs

- **11 samples** (4 x 16 ppi, 3 x 23 ppi, 4 x 33 ppi)
- **Kirschhoff matrix = degree matrix - adjacency matrix**
- **PCA on Kirschhoff matrix eigenvalues**

**Kirschhoff matrix eigenvalues**

**PCA projections**
Convergent toolkit
- Homotopic thinning preserve topology
- Proof of convergence of metric graph reconstruction

Generic C++ implementation
- Arbitrary ambient and embedded dimensions
- DEC module included in DGtal
- Linear algebra solvers from Eigen and ARPACK
- Open source

http://dgtal.org
Typical performance: solving heat flow

IO + discretization + problem construction + factorization + resolution

256^3 px 1358000 $\sigma^0$

300 s 1CPU@2.6GHz