Metric graph model of fibrous material

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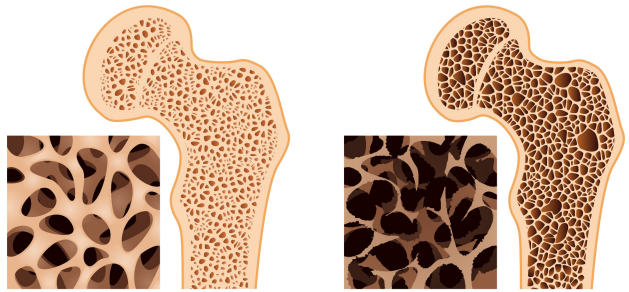
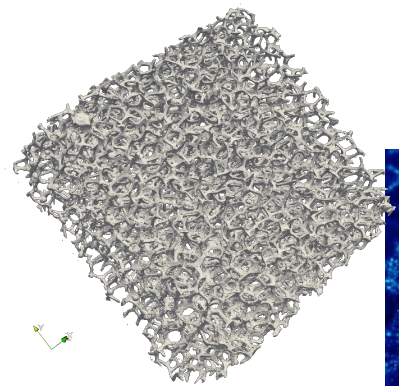
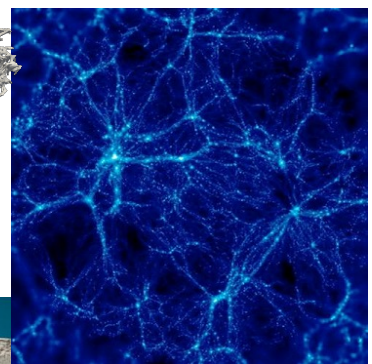
Fibrous materials

Appear in many fields

- Galaxy superclusters
- Reinforced concrete
- Internal bone structure
- Catalytic foam







Fibrous materials

Appear in many fields

- Galaxy superclusters
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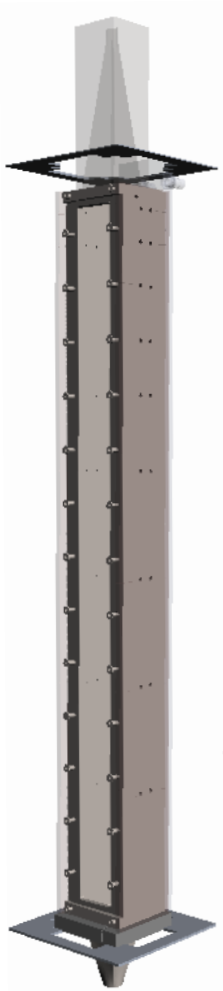
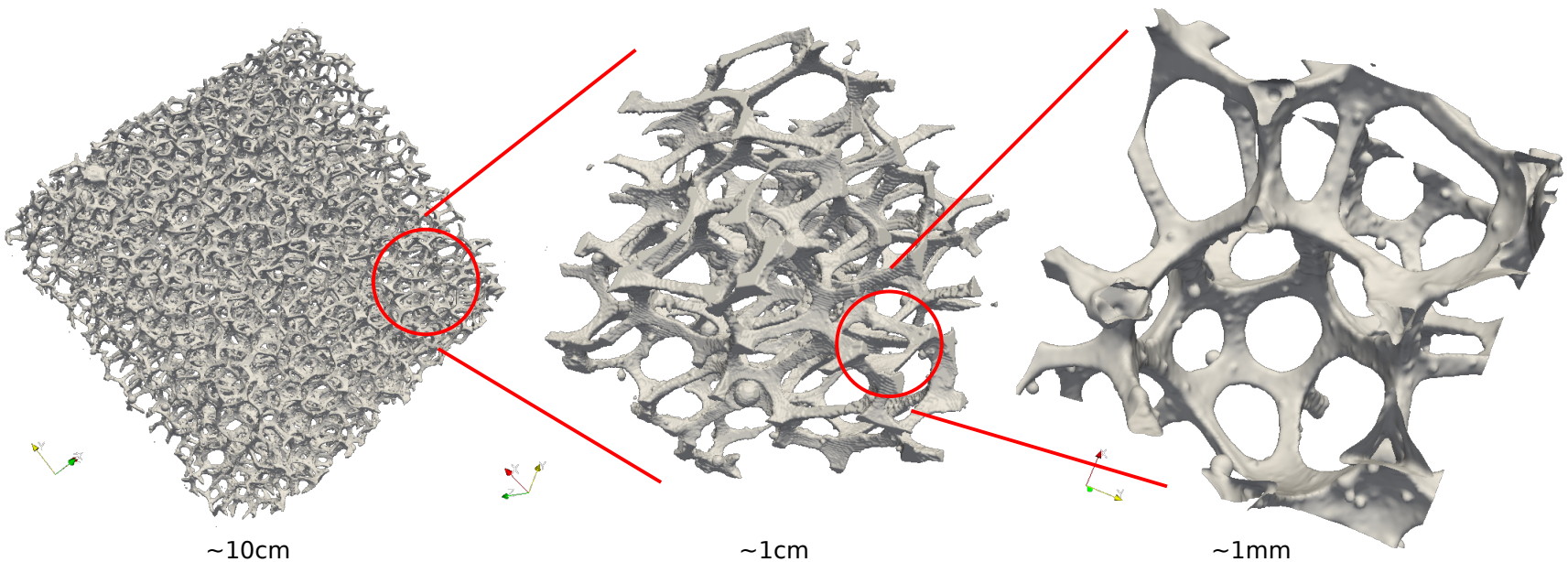
- Gas/gas or gas/solid reactions
- Various materials
  - Metal (Cu, Al)
  - Plastic (Polyurethane)
- Large stiffness to mass ratio
- High surface to volume ratio



Thermal bench (LGPC)
see posters P-70 and P-50

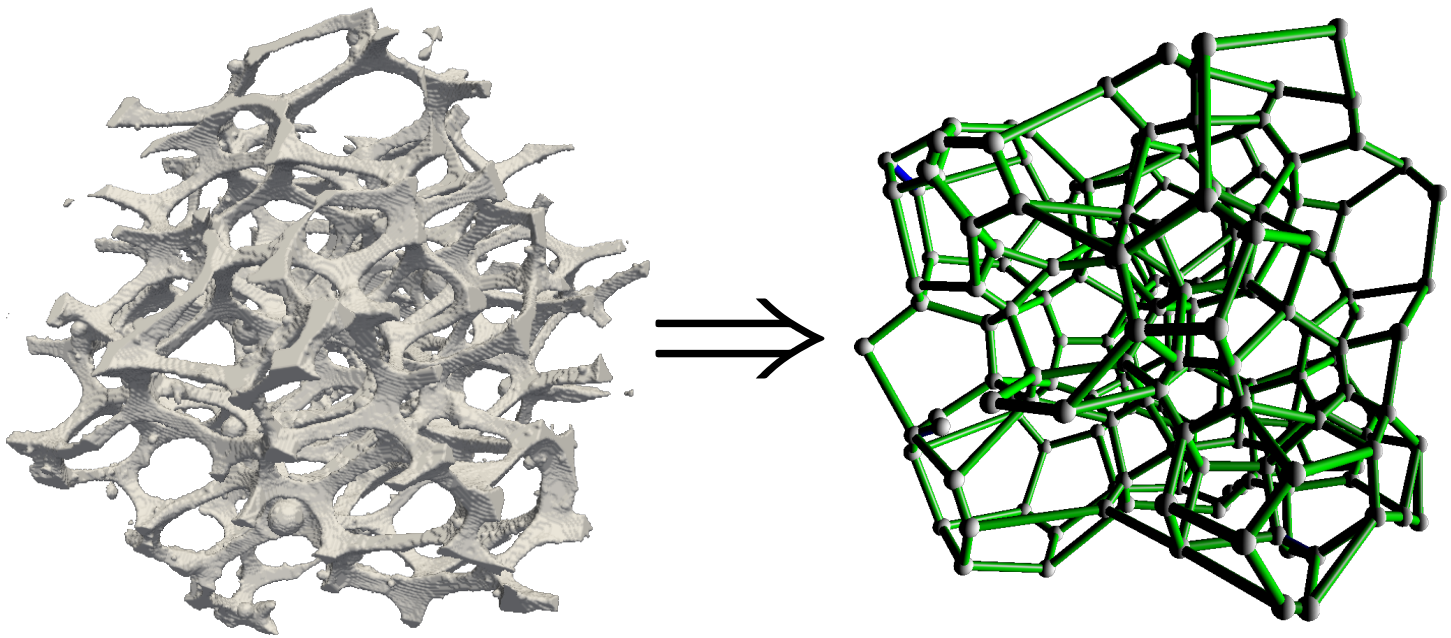
Catalytic foam

- Non repeatable manufacturing process
- Internal inhomogeneities
- Complex non planar structures
- Multiple scales



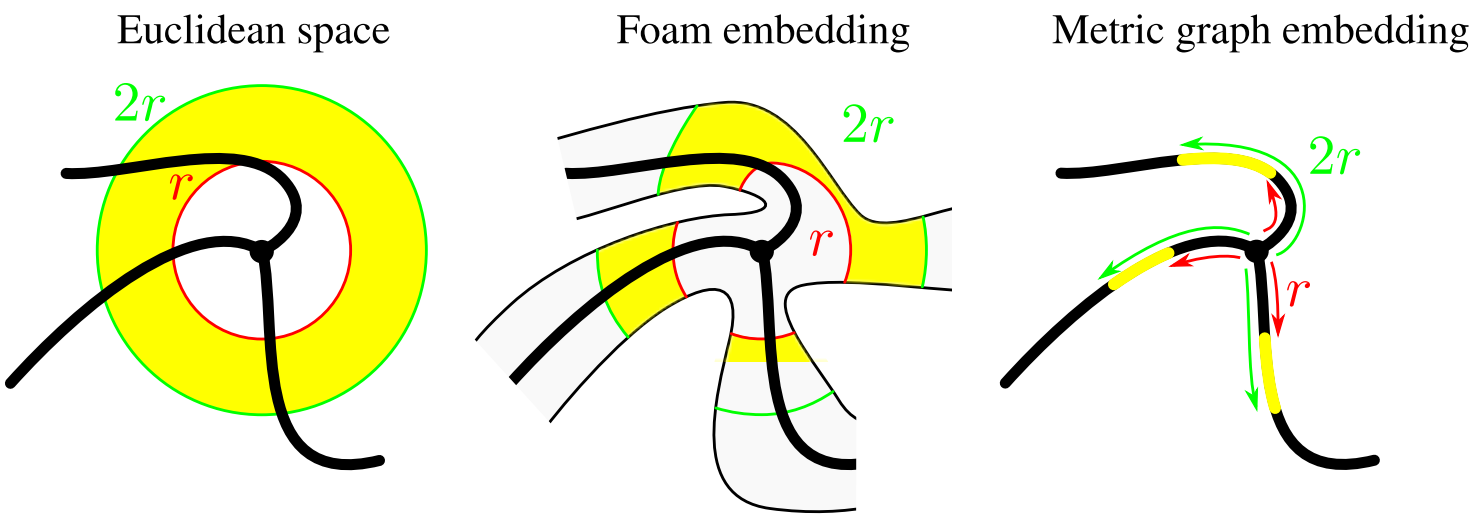
Metric graph model

- Fibrous material properties
  - Topology : cells, holes and cycles
  - Geometry : lengths, areas et volumes
- Metric graph
  - Graph = nodes + edges
  - Metric graph embedding = edge lengths
  - 1D differential manifold



Manifold embedding

- Manifold = smooth curved space with boundaries and holes
- Tensorial linear operators, scalar and vector fields
- Geodesic length = shortest length inside manifold



- Foam operators projected on metric graph
- Linear PDE resolution
  - Heat flow
  - Electrical resistivity
  - Mechanical stress

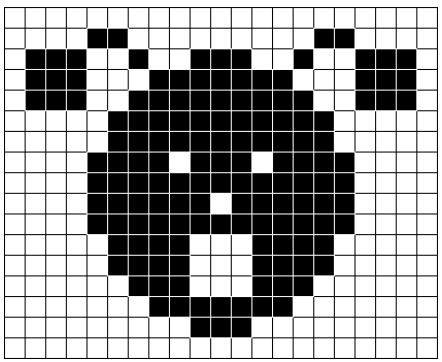
Proposed toolkit

- Morphological closing
- Homotopic thinning
  - Iterative topological skeleton simplification
  - Distance map driven
  - Topology conservation / size reduction
- Discrete Exterior Calculus (DEC)
  - Modern versatile discretization scheme
  - Classical linear operators ∇, Δ, $\nabla\cdot$ and $\nabla\wedge$
  - Stokes theorem holds $\Rightarrow$ unique convergent solution
  - Khalimsky space $\Rightarrow$ trivial well conditioned discretization
- Geodesic in heat *[CRANE13]*
- Metric graph reconstruction *[AANJANEYA12]*

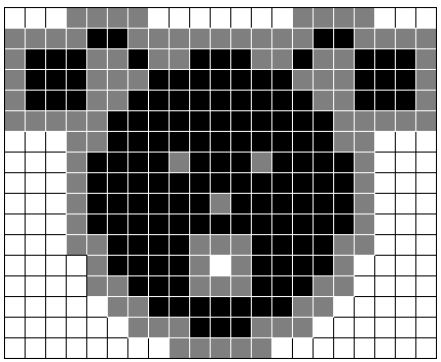
Mathematical morphology

- Close hollow tube
- Noise reduction after thresholding

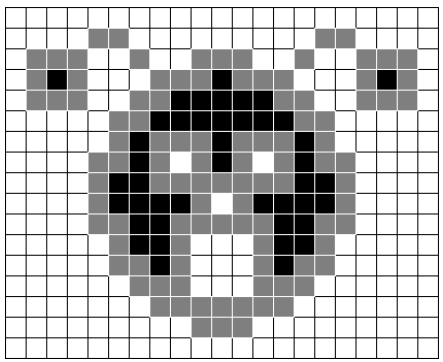
Original



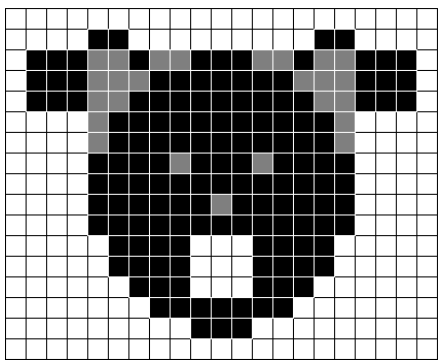
Dilation



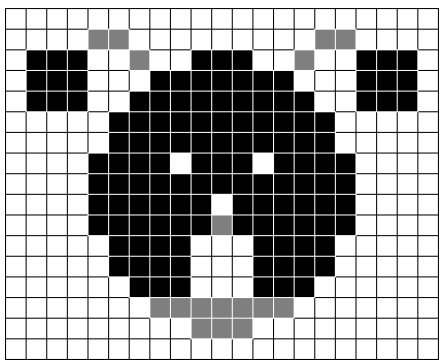
Erosion



Closing = Erosion ∘ Dilation



Opening = Dilation ∘ Erosion



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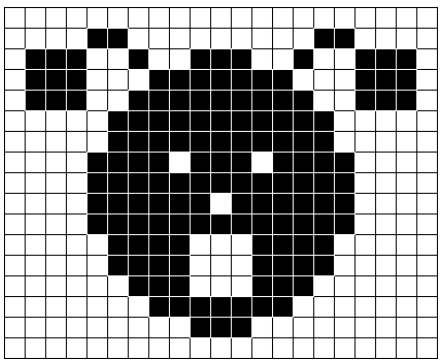

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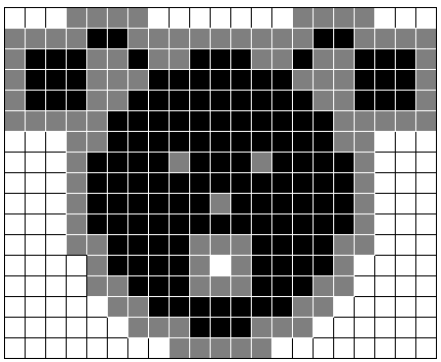

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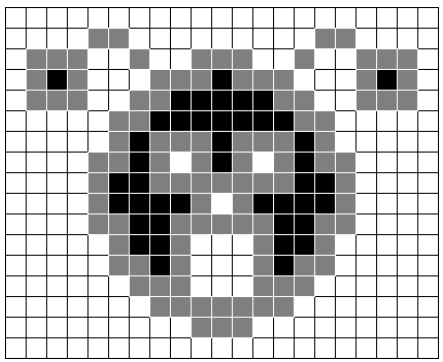

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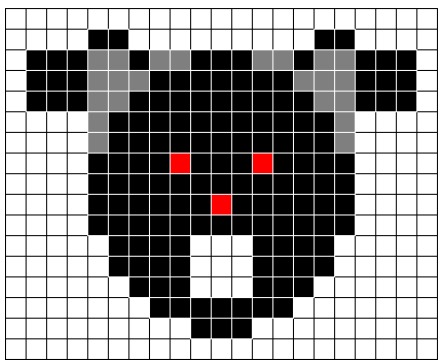
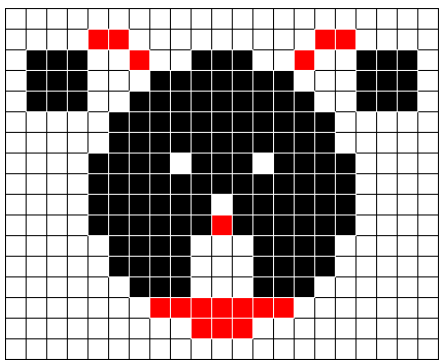


Opening = Dilation ∘ Erosion



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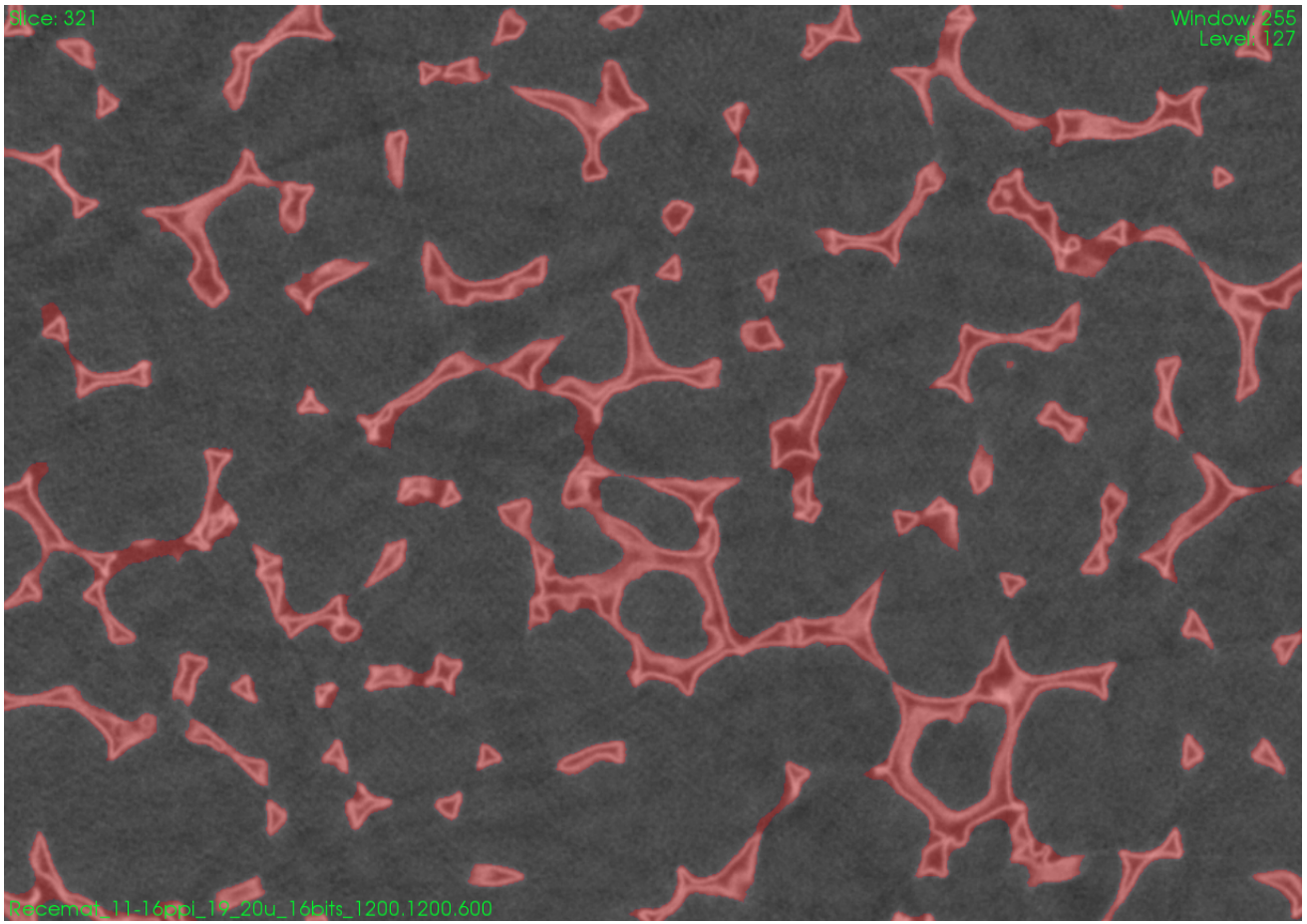
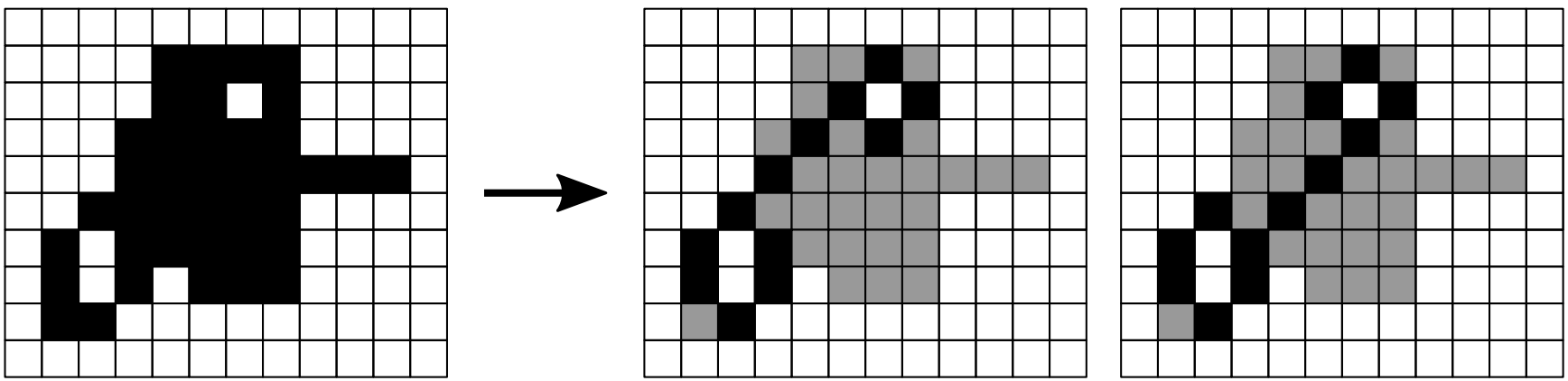


Homotopic thinning

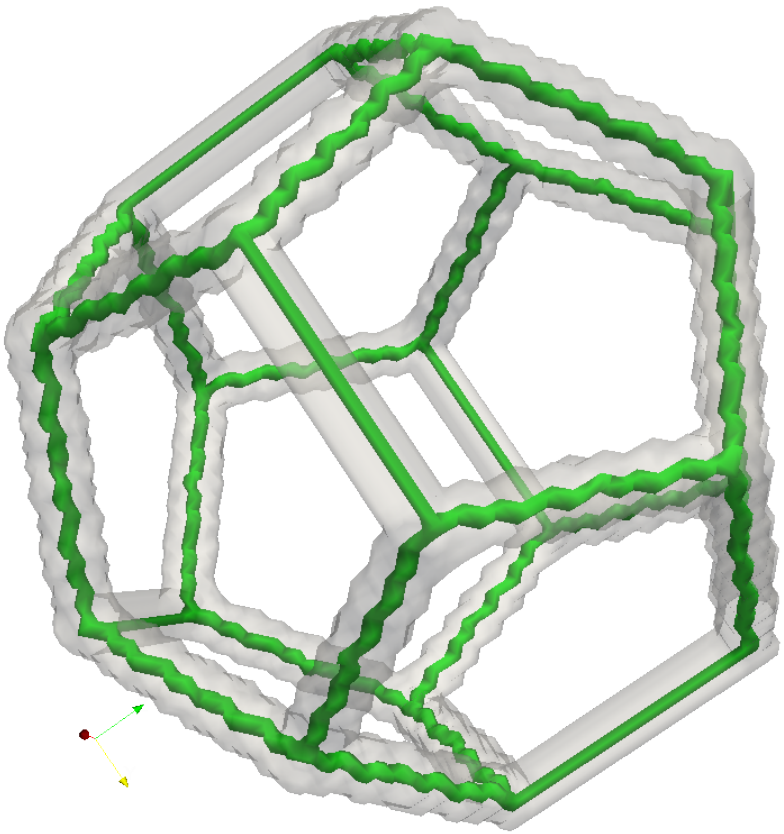
Size reduction with topology conservation

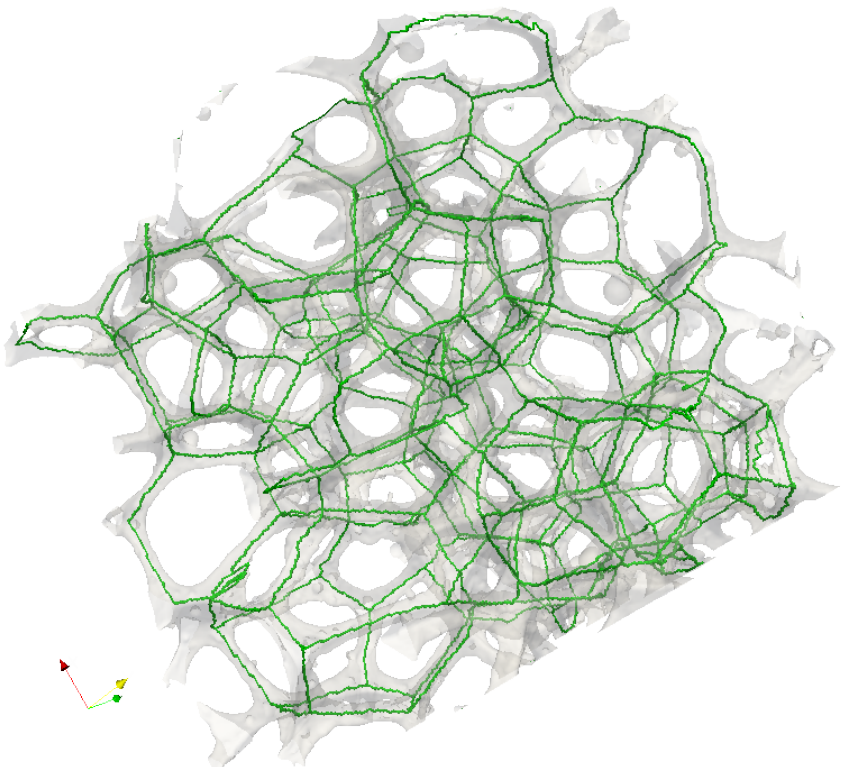
- Iterative erosion process
- No hole filling or closing
- Non unique solution



- Local simplicity test ⇒ efficient (LUT)
- Distance weighed priority queue ⇒ keep skeleton centered

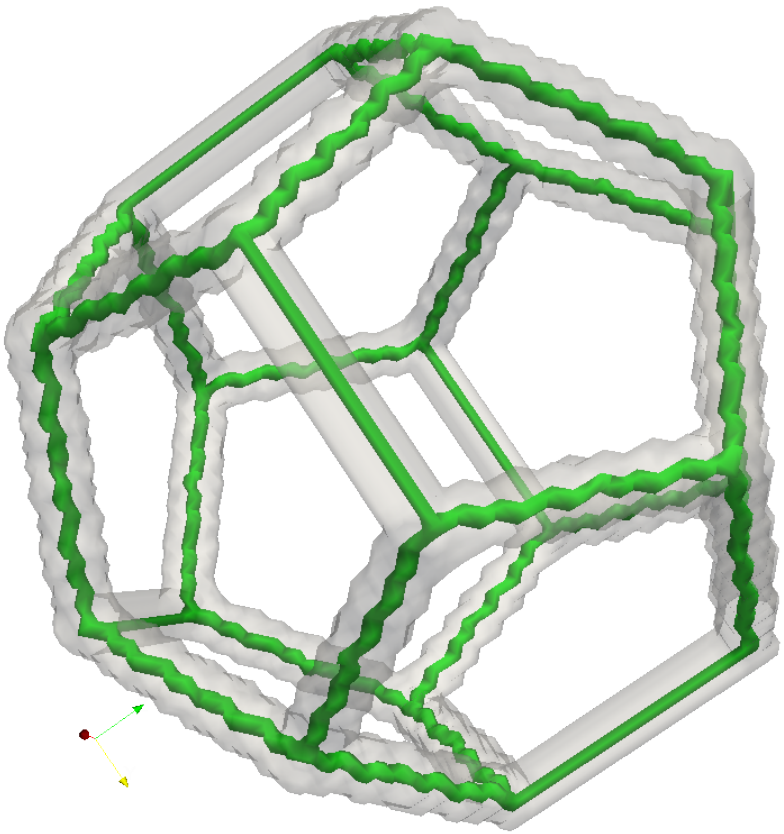
Homotopic thinning

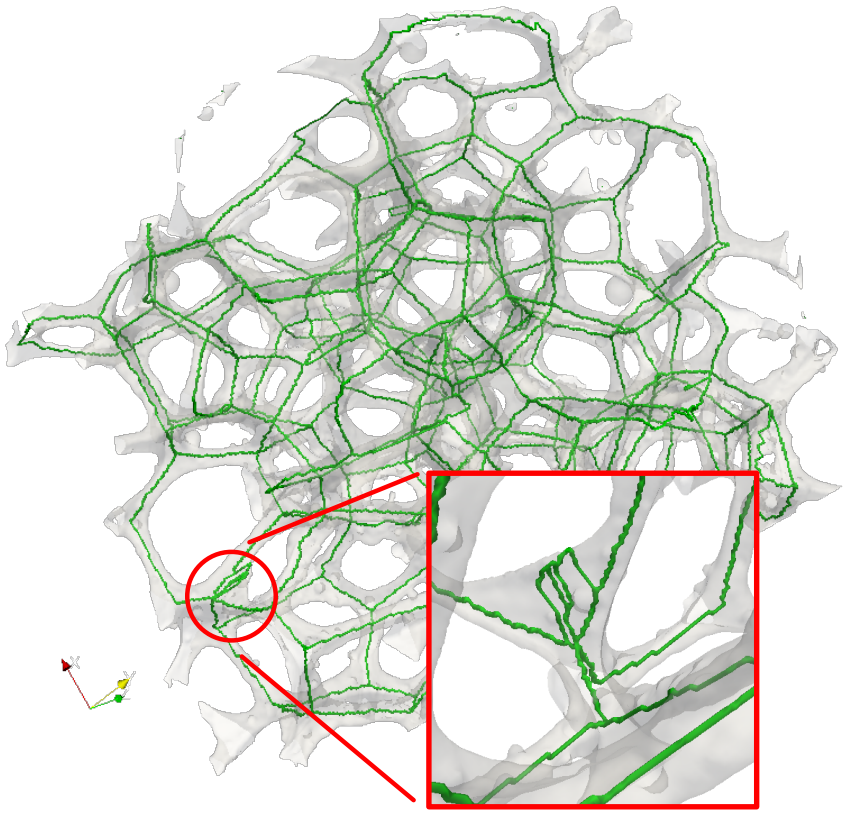




Homotopic thinning

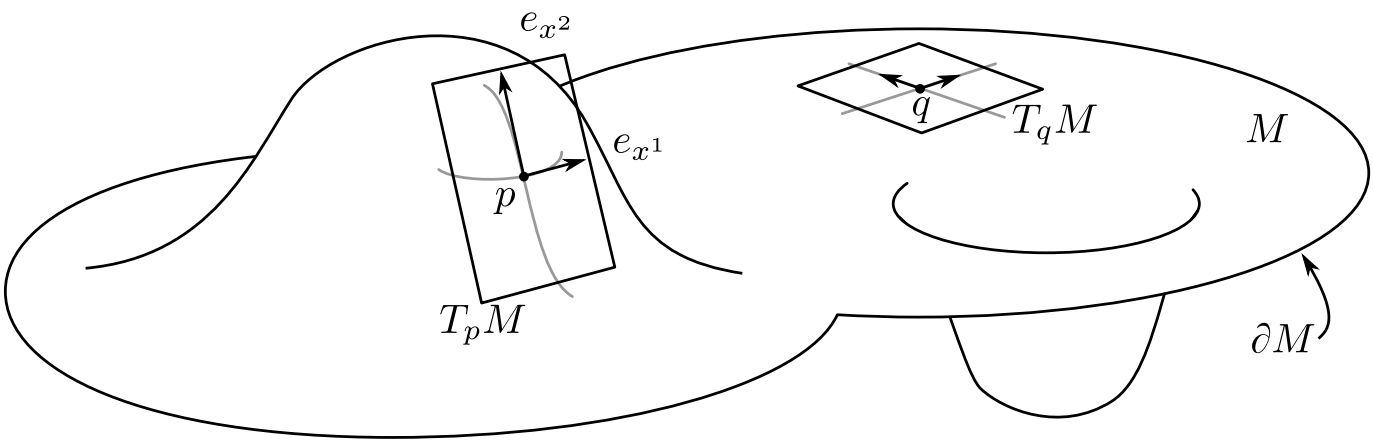
May be geometrically inaccurate





Continuous Riemannian manifold

- Smooth curved space M with boundary ∂M
- Points p, q
- Coordinates system x^1, x^2, ...
- 0-form $\Leftrightarrow$ scalar field
- 1-form $\Leftrightarrow$ vector field $\in T_pM$ (local tangent vector space)



Base operators

$\ast$ Hodge (orthogonality) $\wedge$ wedge (span)

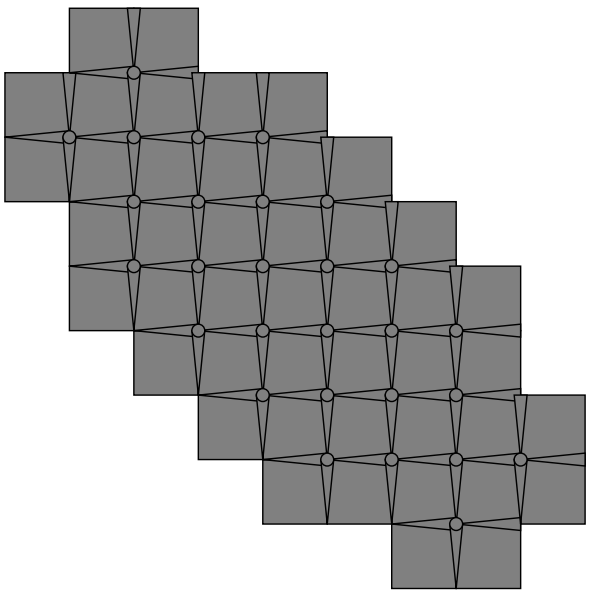
d exterior derivative

Discrete exterior calculus (DEC)

- Cellular complex (with dual)
- 0-cells $\Leftrightarrow$ point, 1-cells $\Leftrightarrow$ edge, ...
- k-cells holds discrete k-form values
- Khalimsky space
- Convergent operators

Stokes' theorem

$$\int_M d\omega = \int_{\partial M} \omega$$

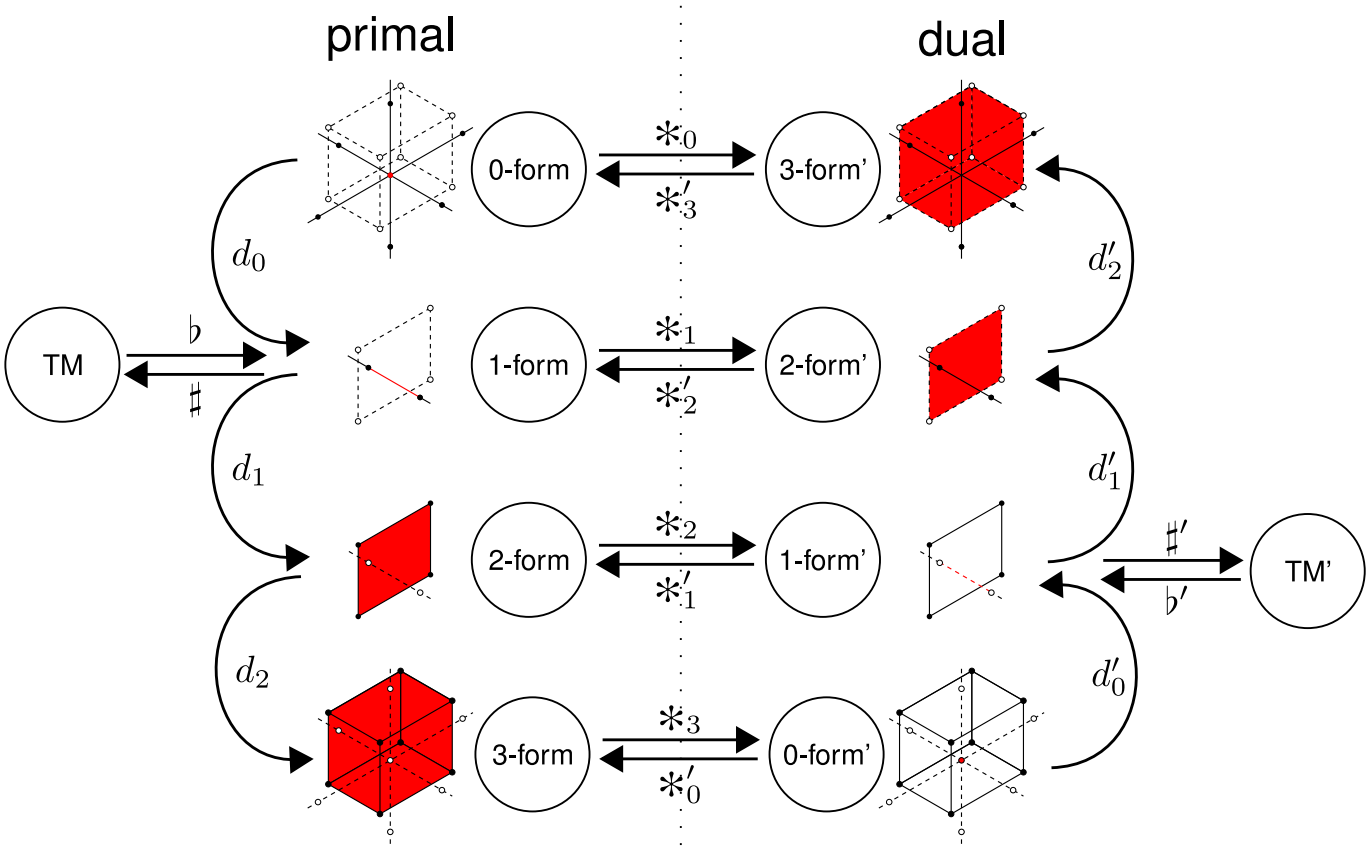


Khalimsky cellullar complex

Classical operators

Gradient	$\nabla = d$
Divergence	$\nabla\cdot = *d*$
Curl	$\nabla\wedge = d*$
Laplacian	$\Delta = *d*d$

Operators chain and cochain



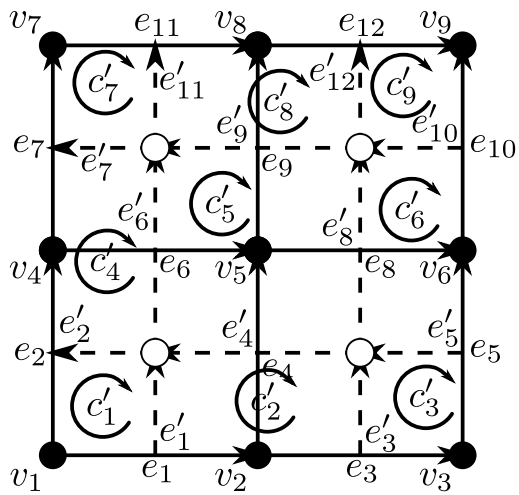
f scalar field (0-form) V vector field (1-form$^\sharp$)

$$\nabla f = (d_0 f)^\sharp \qquad \nabla \cdot V = *'_2 d'_1 *_1 V^\flat$$

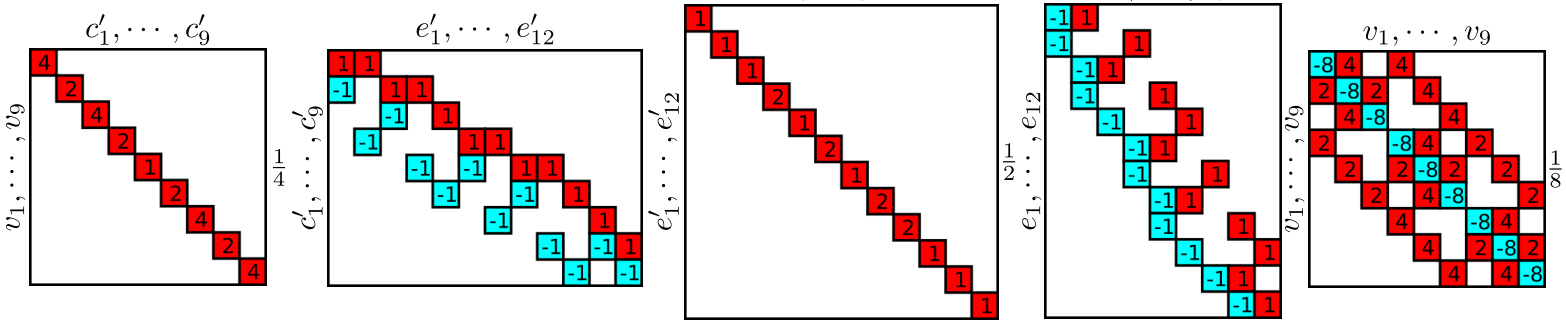
$$\Delta f = \nabla \cdot \nabla f = *'_2 d'_1 *_1 d_0 f \qquad \nabla \wedge V = (*_2 d_1 (V)^\flat)^{\sharp'}$$

Linear differential equation ⇒ Linear algebra problem

- Discretization of forms ⇒ discretization of linear operators
- k-form ∼ vector
- Operator ∼ matrix
- Composition ∼ matrix multiplication
- Induction of properties (SDP, ...)



$*'_2$ d'_1 $*_1$ $d_0 = \Delta$



Geodesic in heat *[CRANE13]*

1. Heat flow
   $\frac{\partial u}{\partial t} = \Delta u$
   $t \ll 1$
   $u(t = 0) = u_0$
2. ∇ normalization
   $X = \frac{\nabla u}{|\nabla u|}$
3. Poisson
   $\Delta \phi = \nabla \cdot X$
4. $u_0 = \delta \Rightarrow$
   ϕ distance map

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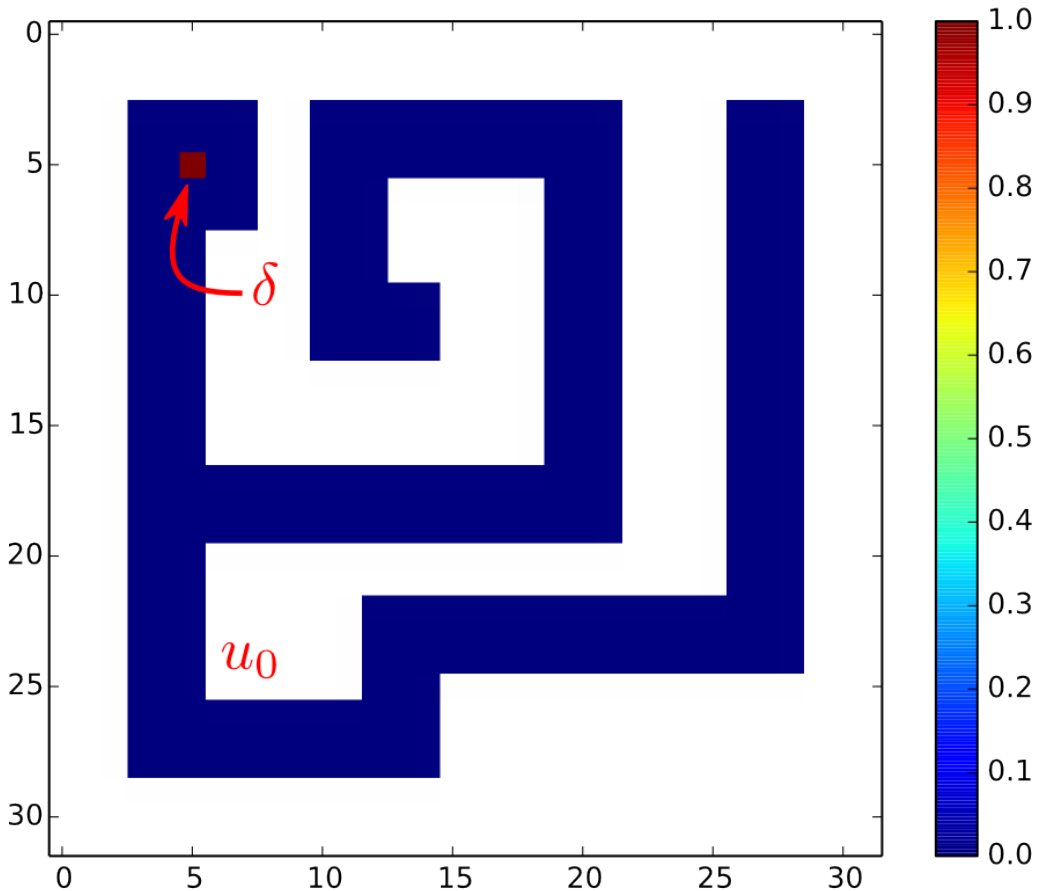

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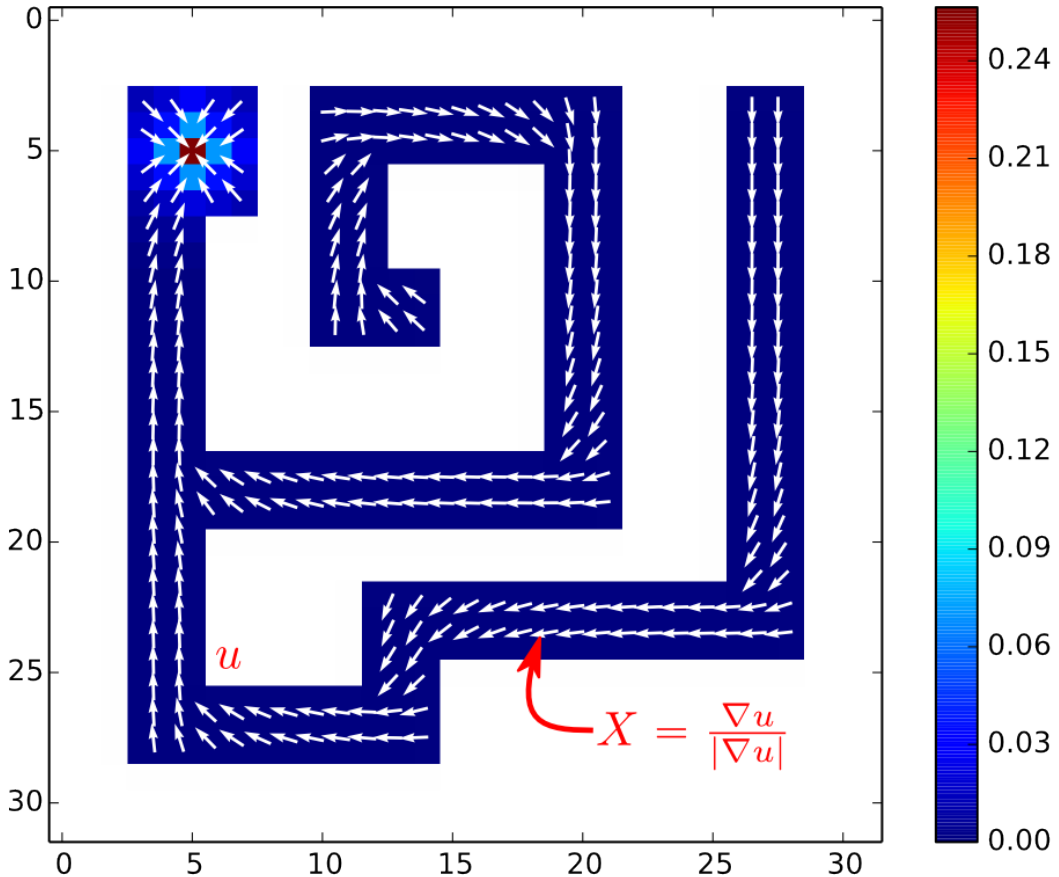

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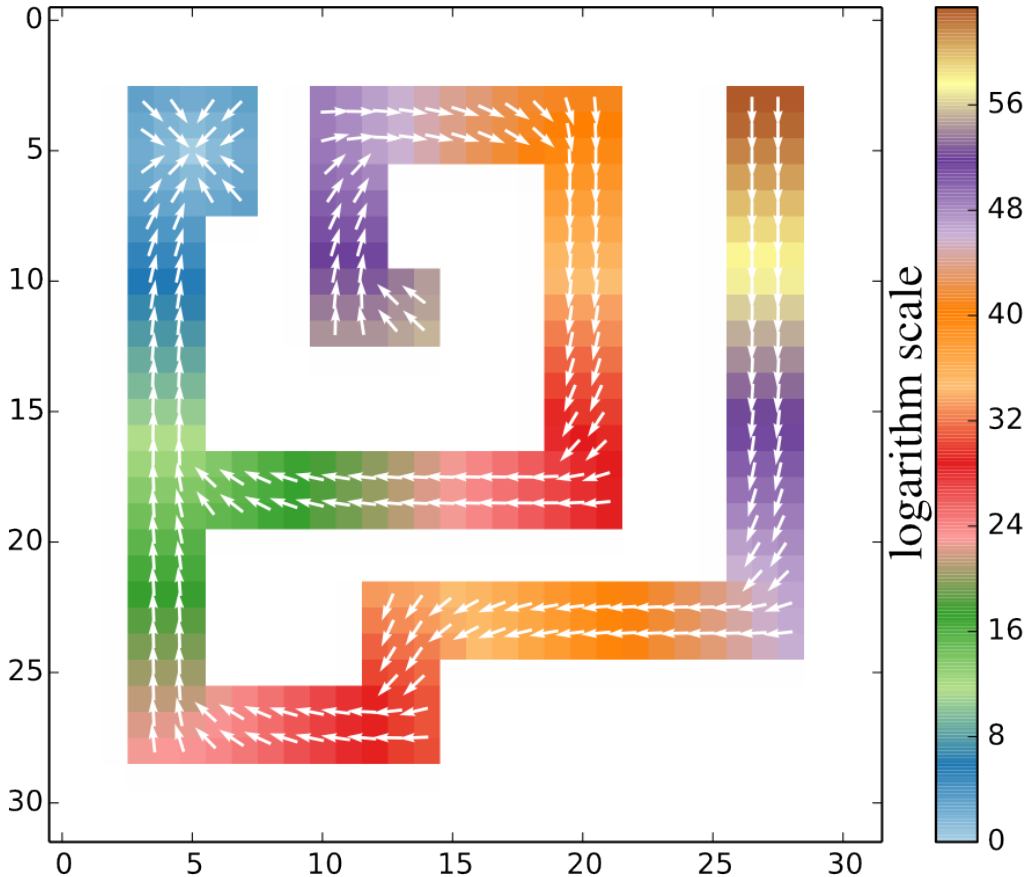

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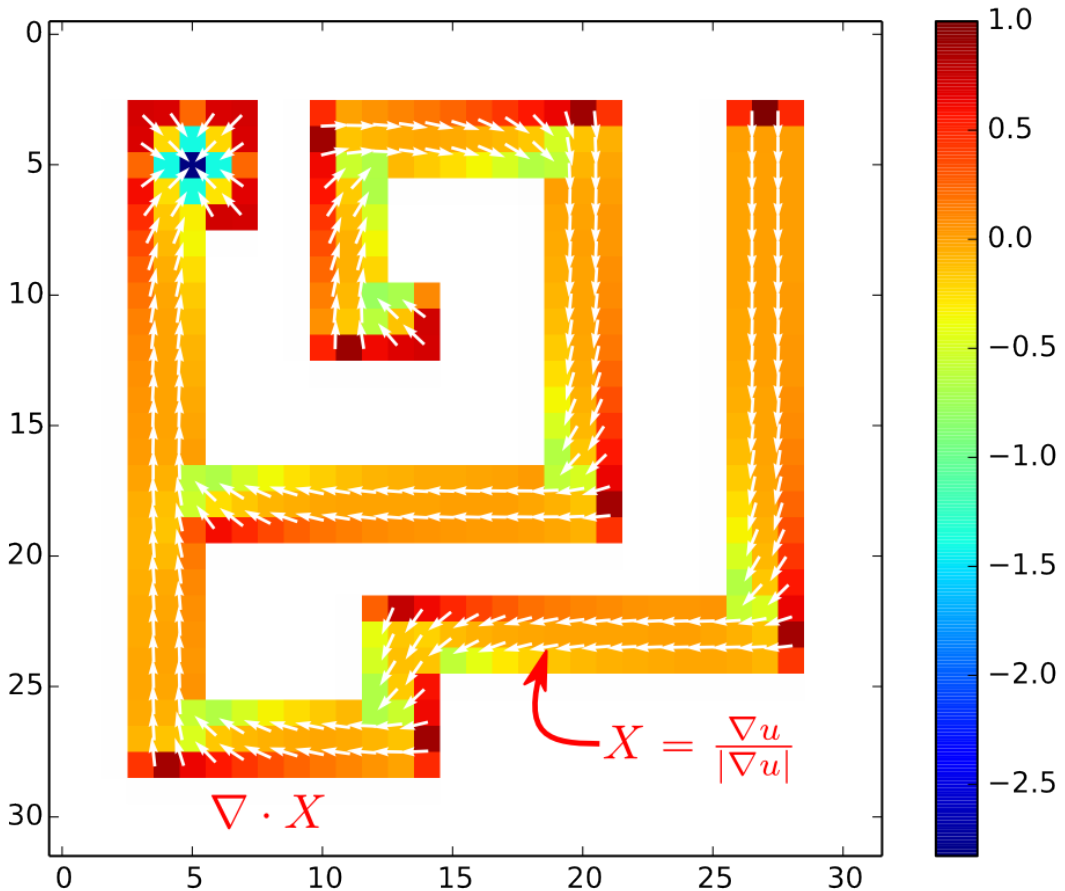

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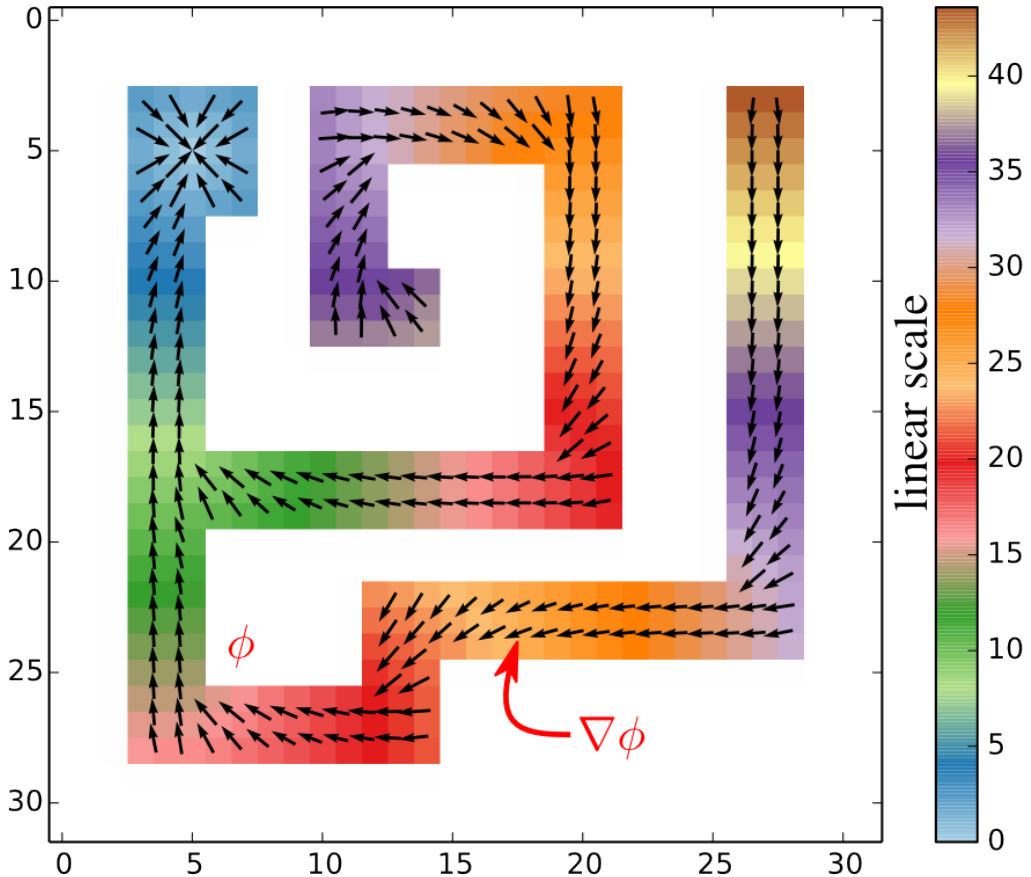

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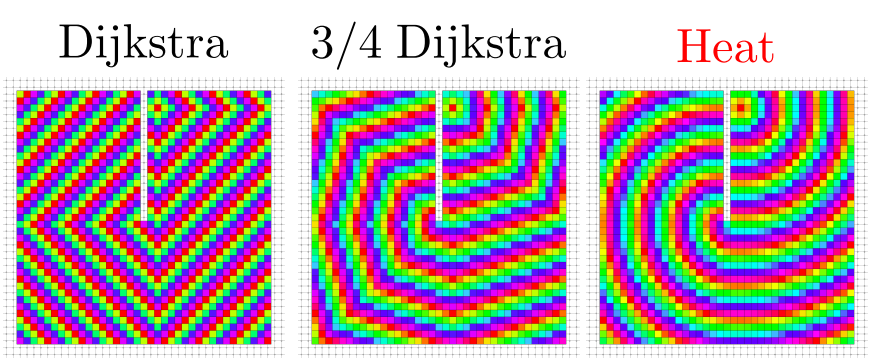
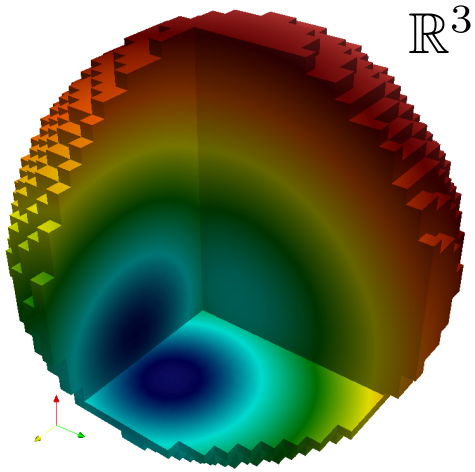
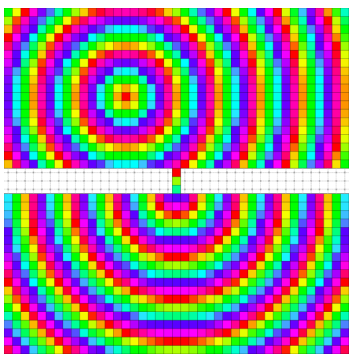
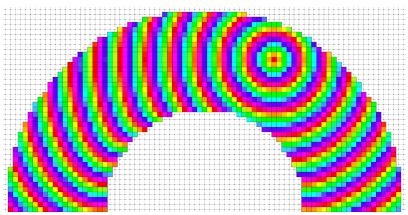
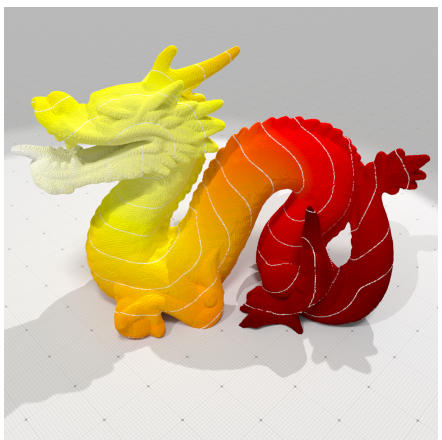


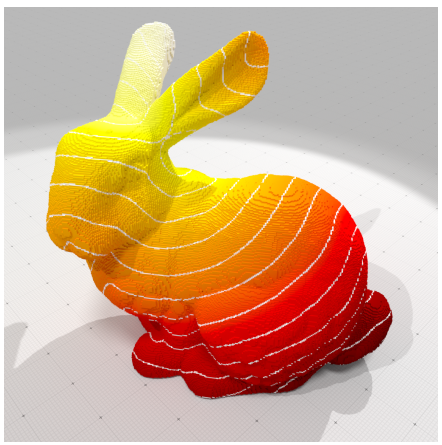




$\mathbb{R}^2$



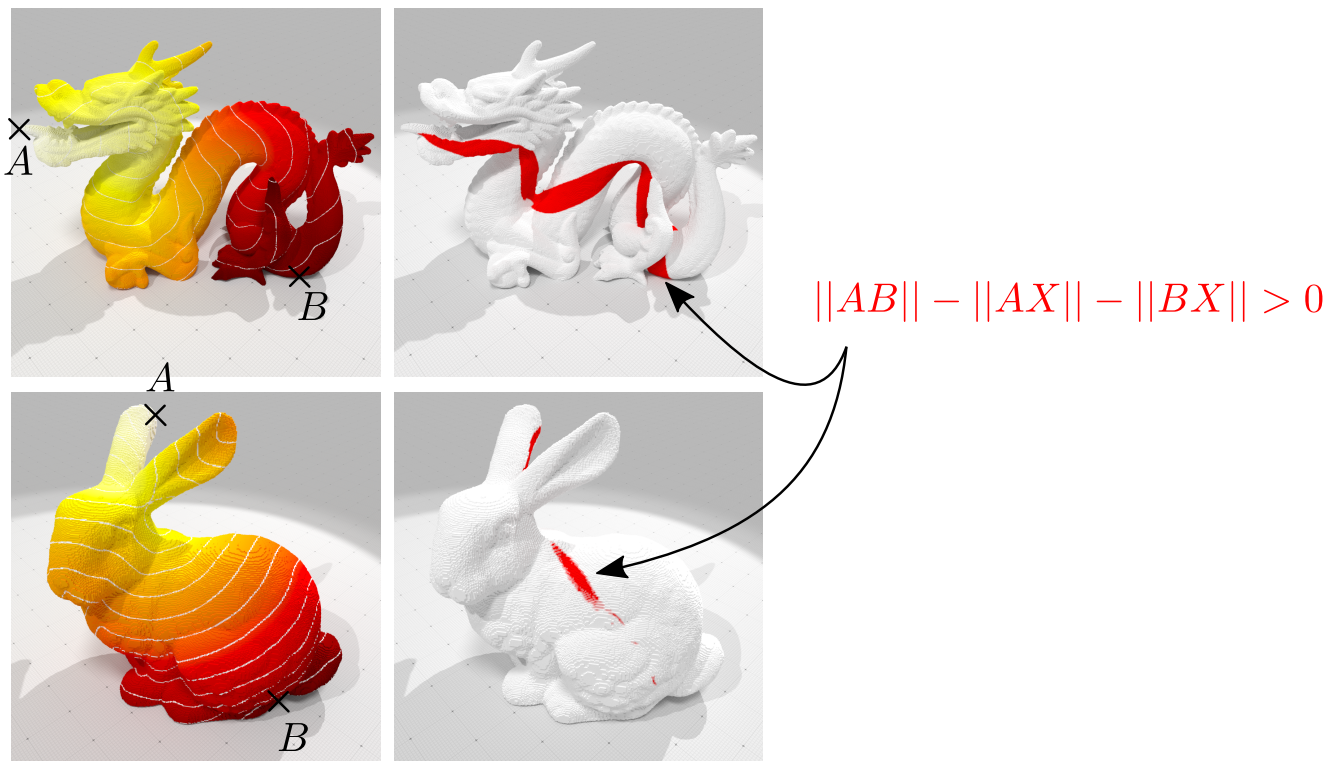




$M \subset \mathbb{R}^3$

Geodesic in heat *[CRANE13]*

- Fast (prefactored linear problem)
- Precise (absolute error $\sim 0 - 3$ px)
- Triangular inequality may not hold everywhere

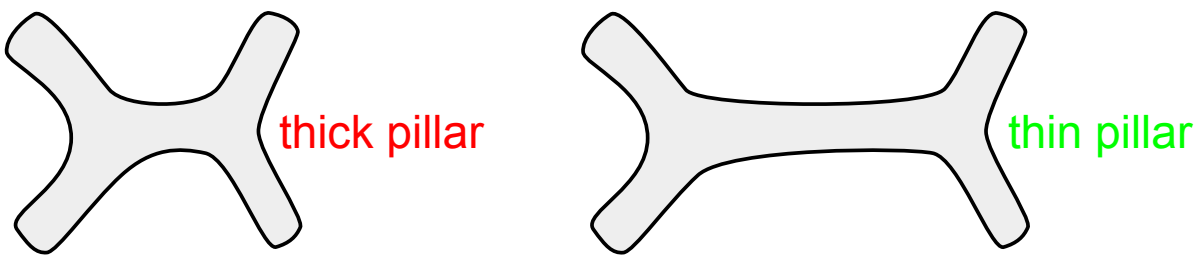


Metric graph reconstruction *[AANJANEYA12]*

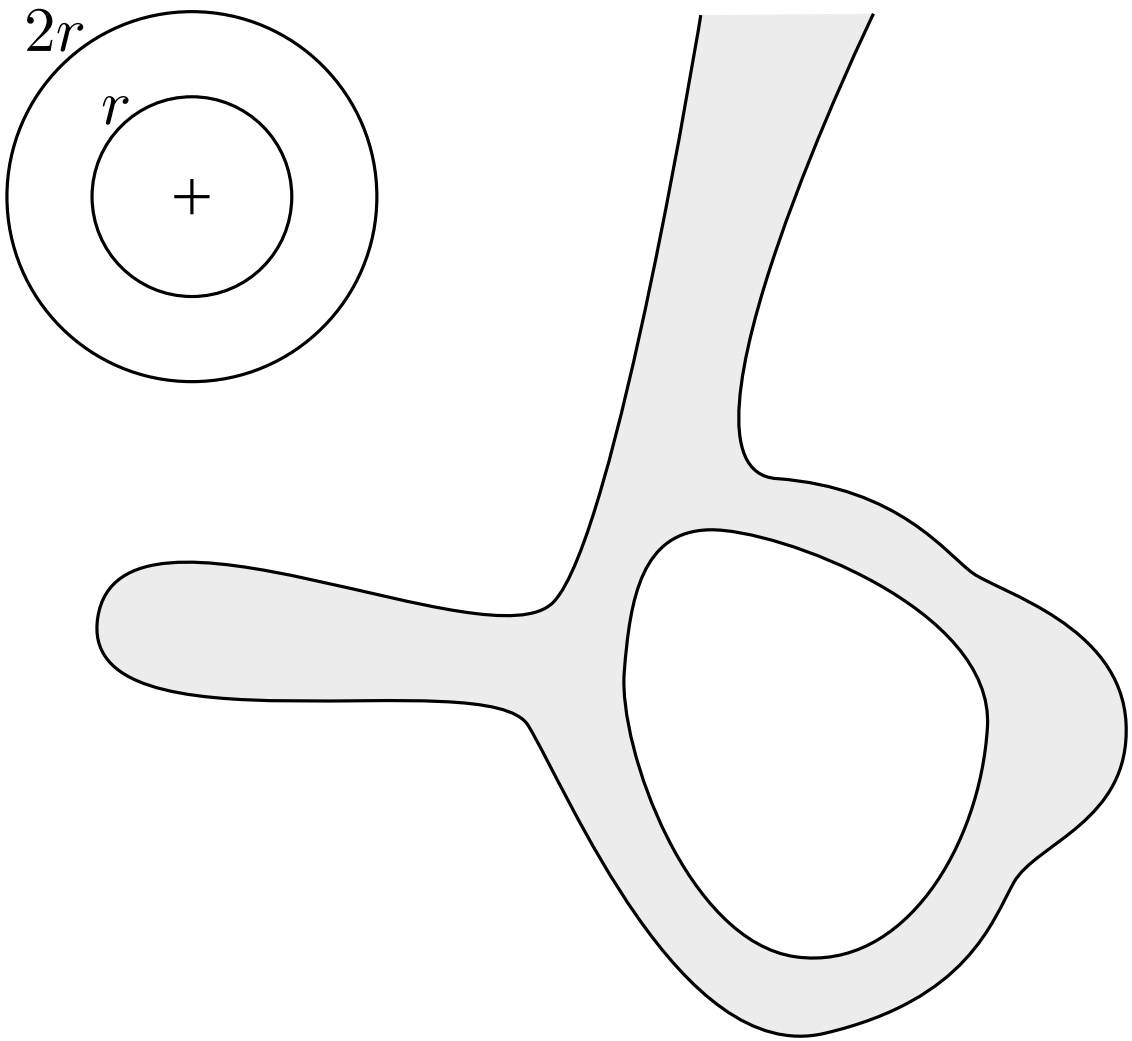
- Degree estimation at each point of the digital

  Degree $=$ # connected components in spherical cap

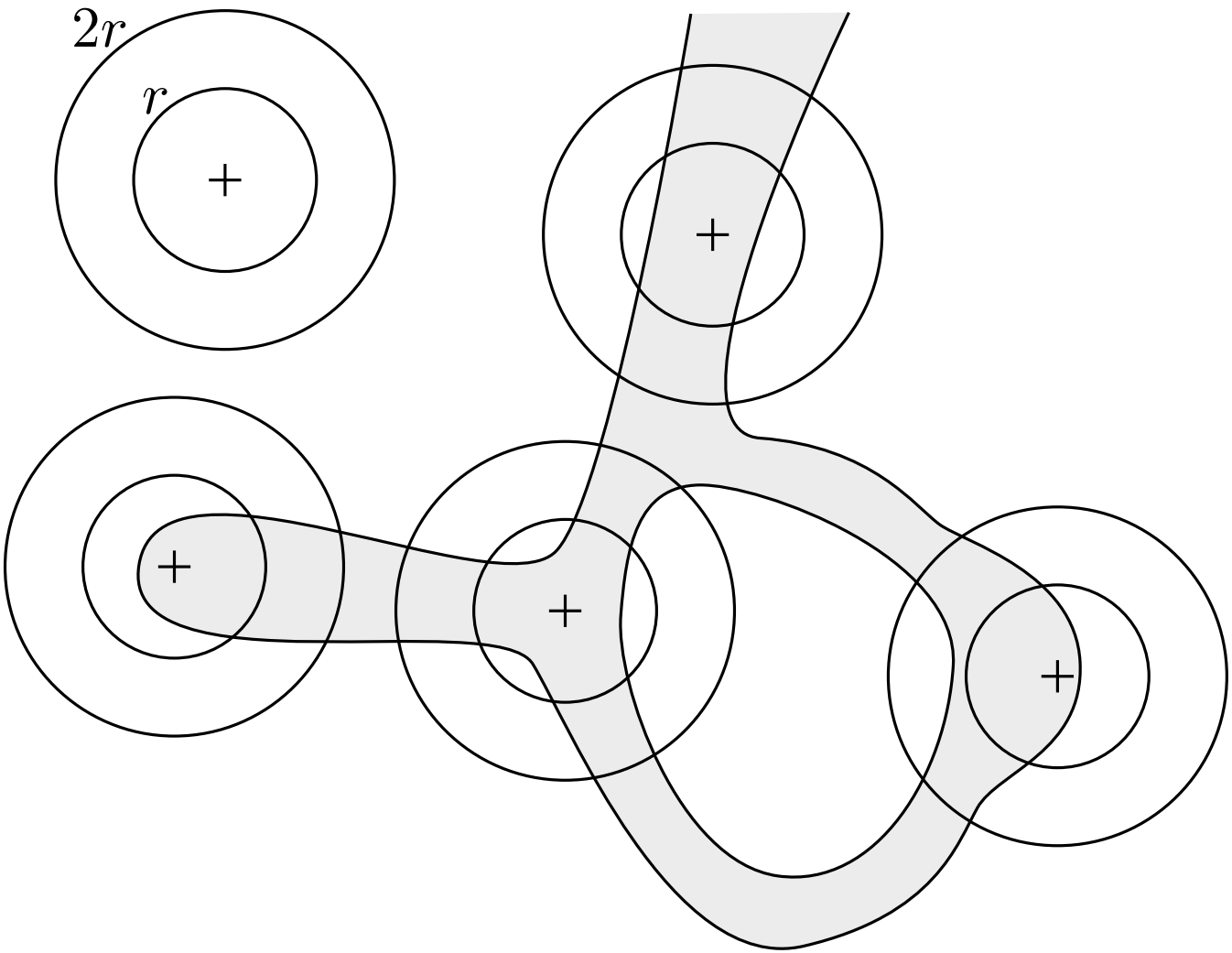
- Spherical caps built using geodesic distance
- Constant degree cluster stands for metric graph elements
  - degree $= 0 \Leftrightarrow$ isolated node
  - degree $= 1 \Leftrightarrow$ dangling edge node
  - degree $= 2 \Leftrightarrow$ edge
  - degree $> 2 \Leftrightarrow$ node
- Proof of convergence if pillars are thin enough



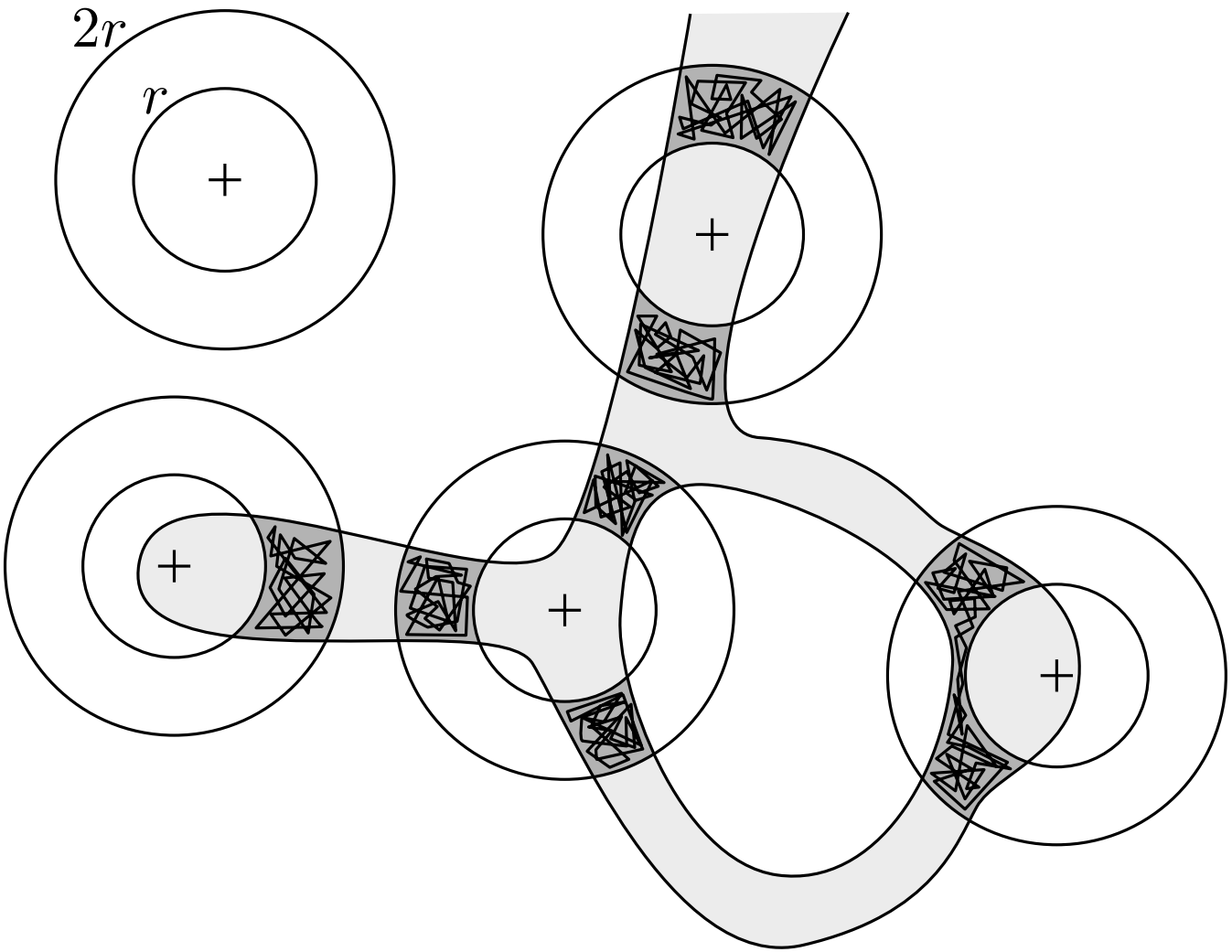
Metric graph reconstruction *[AANJANEYA12]*



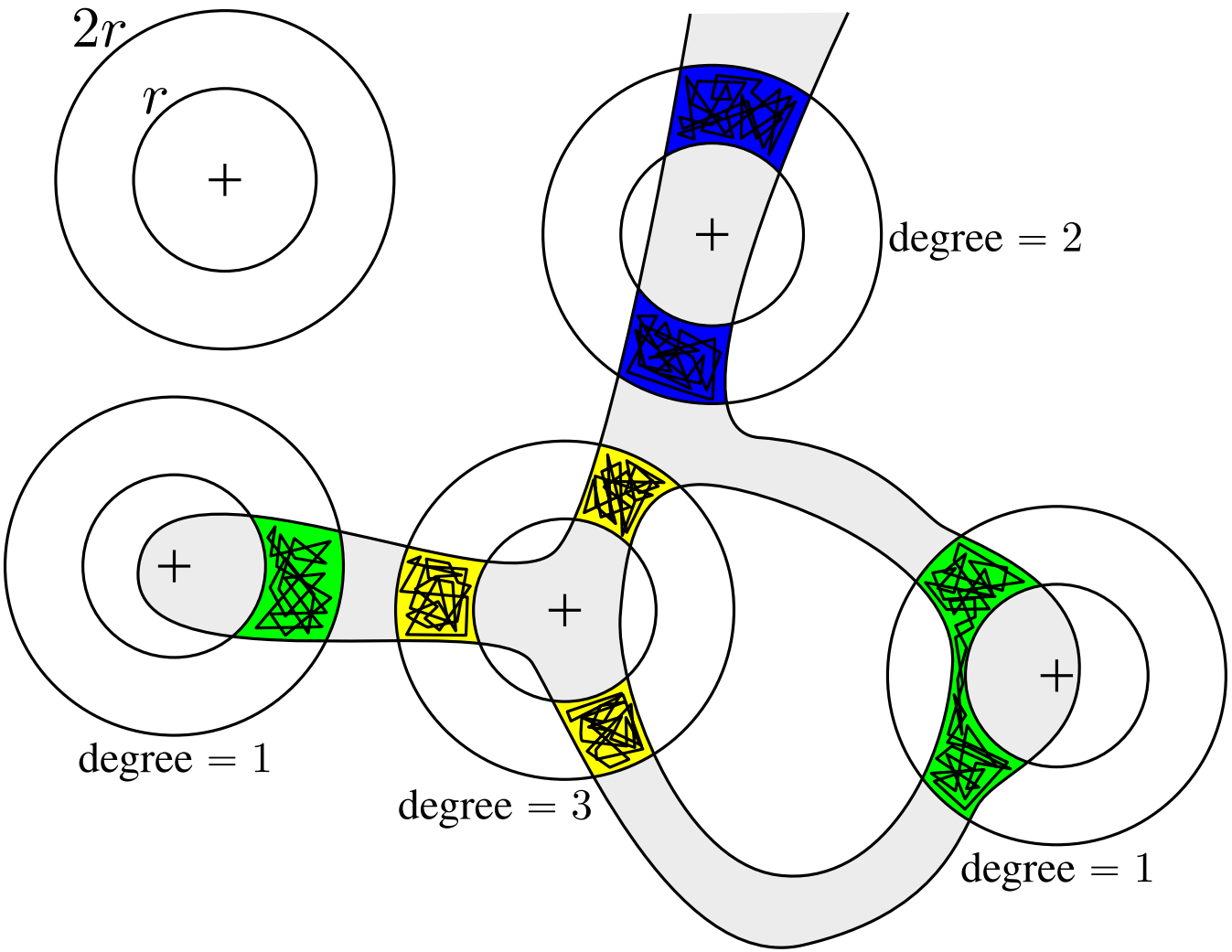
Metric graph reconstruction *[AANJANEYA12]*



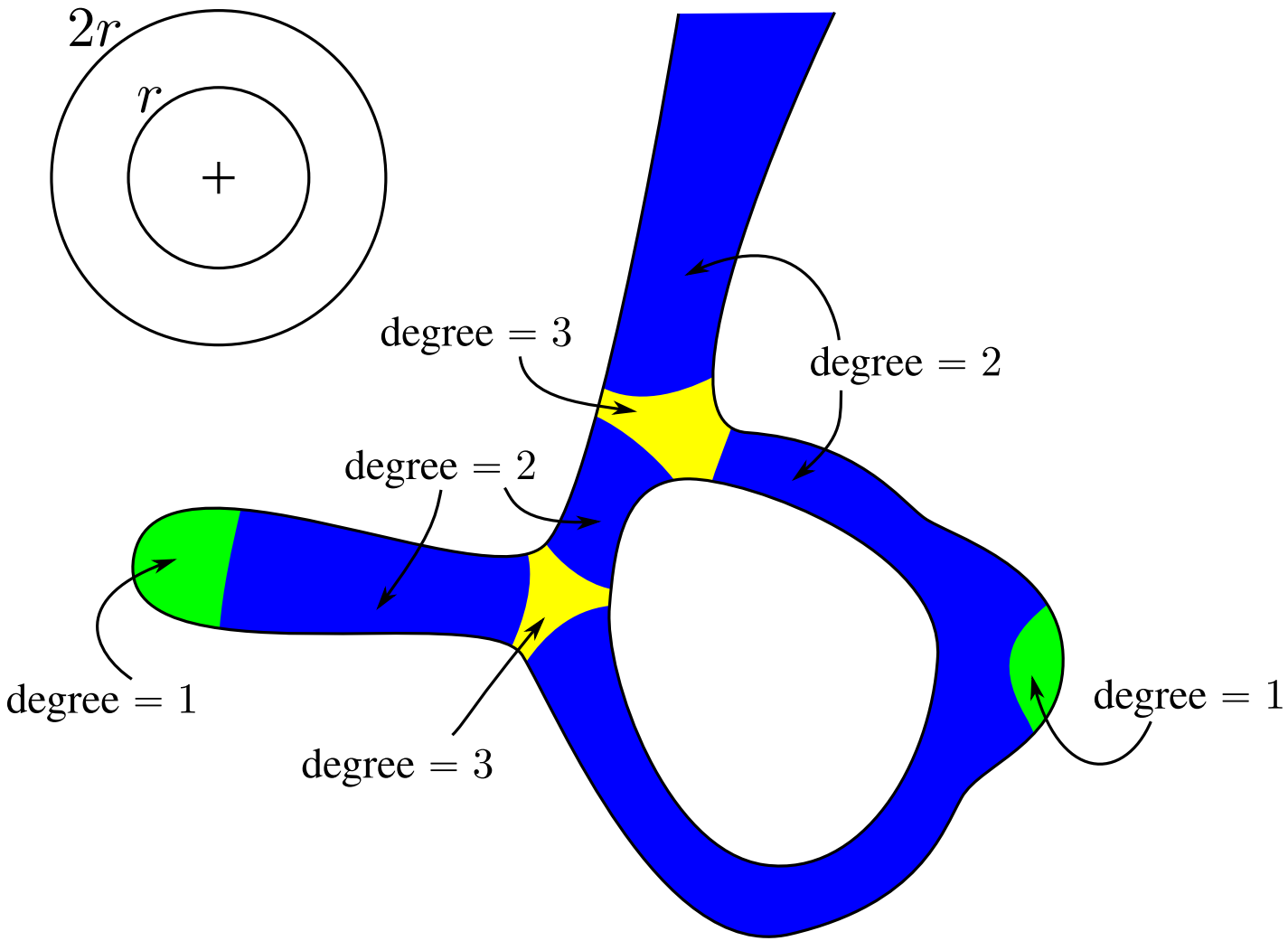
Metric graph reconstruction *[AANJANEYA12]*



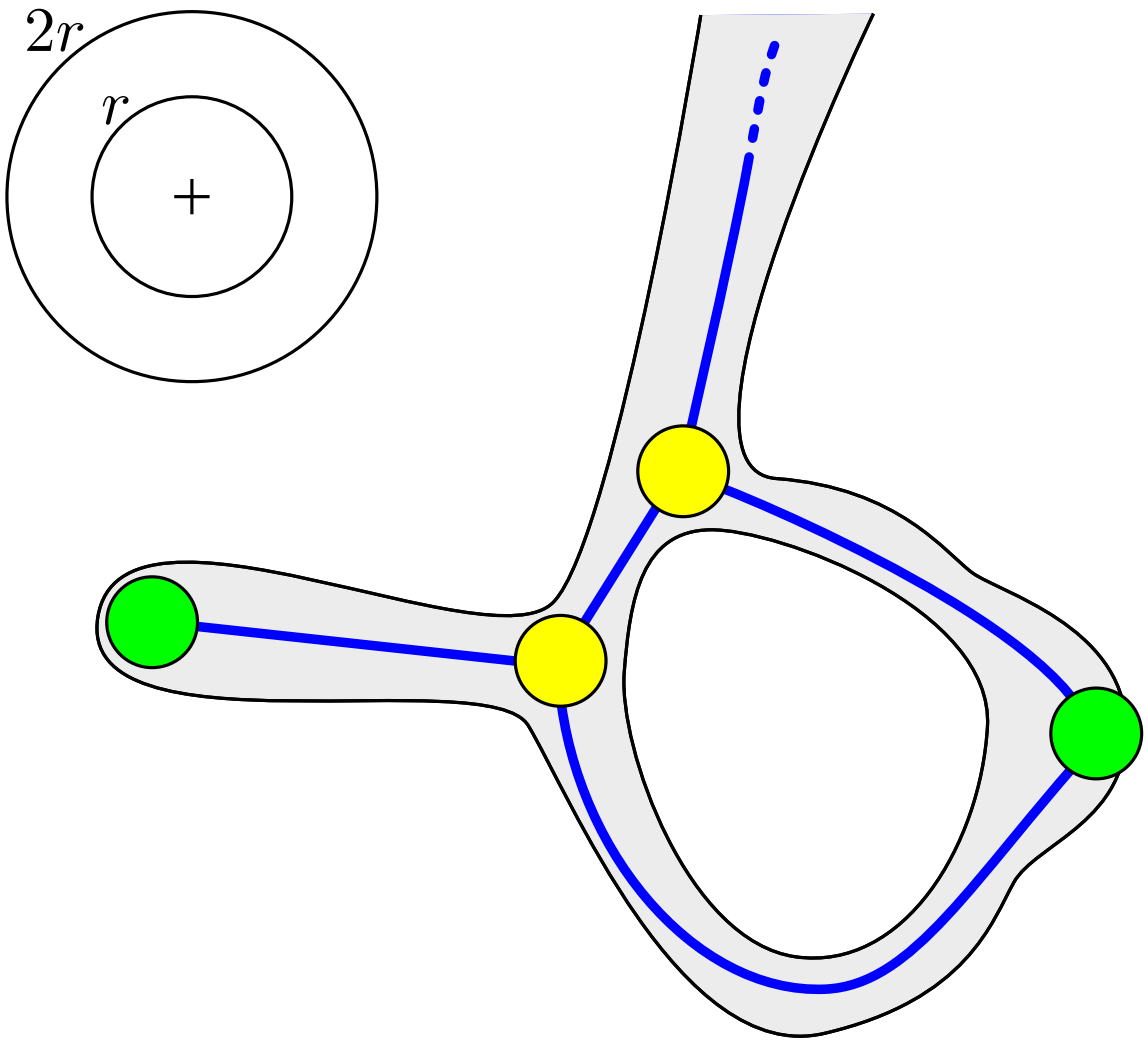
Metric graph reconstruction *[AANJANEYA12]*



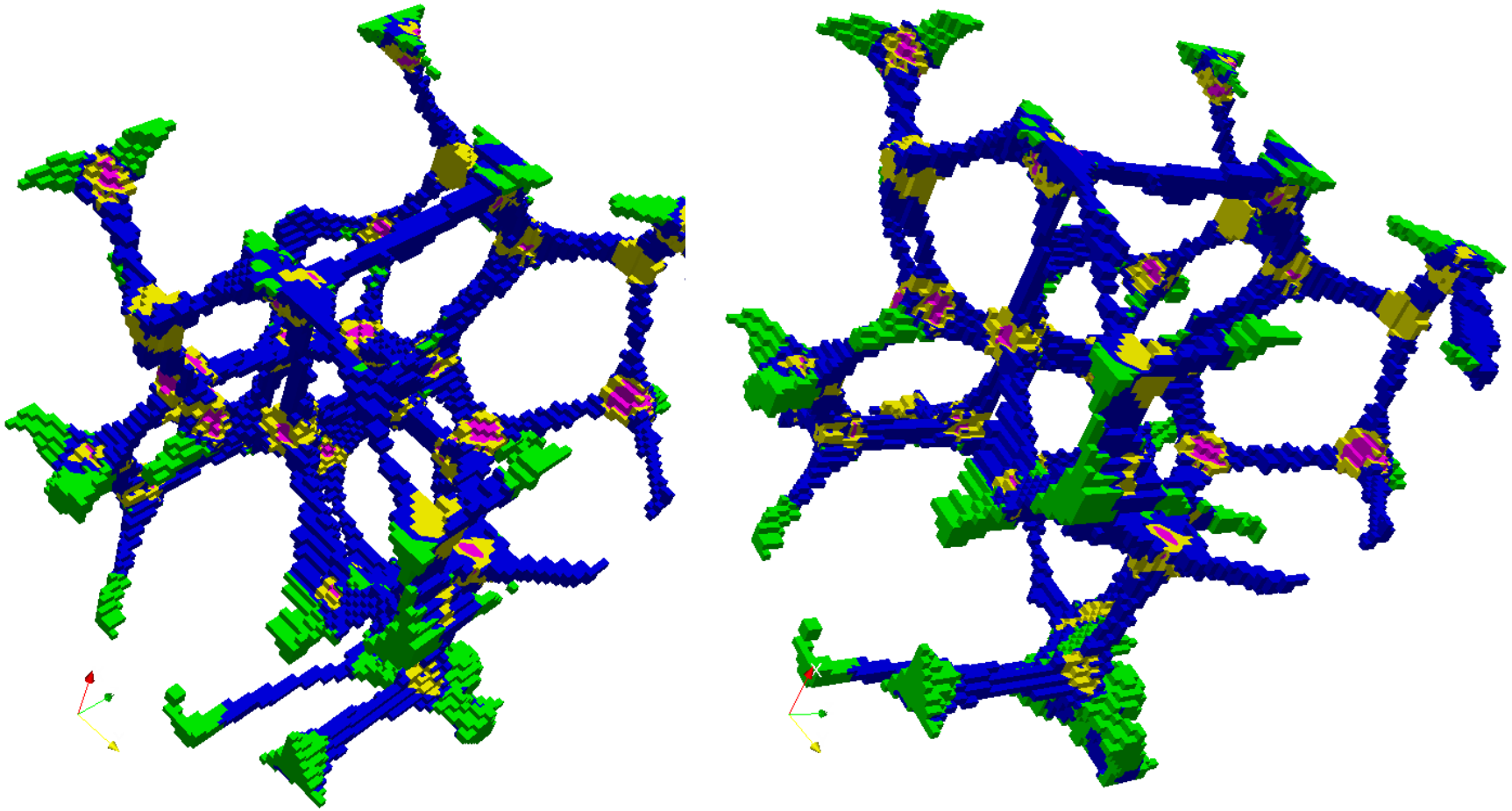
Metric graph reconstruction *[AANJANEYA12]*



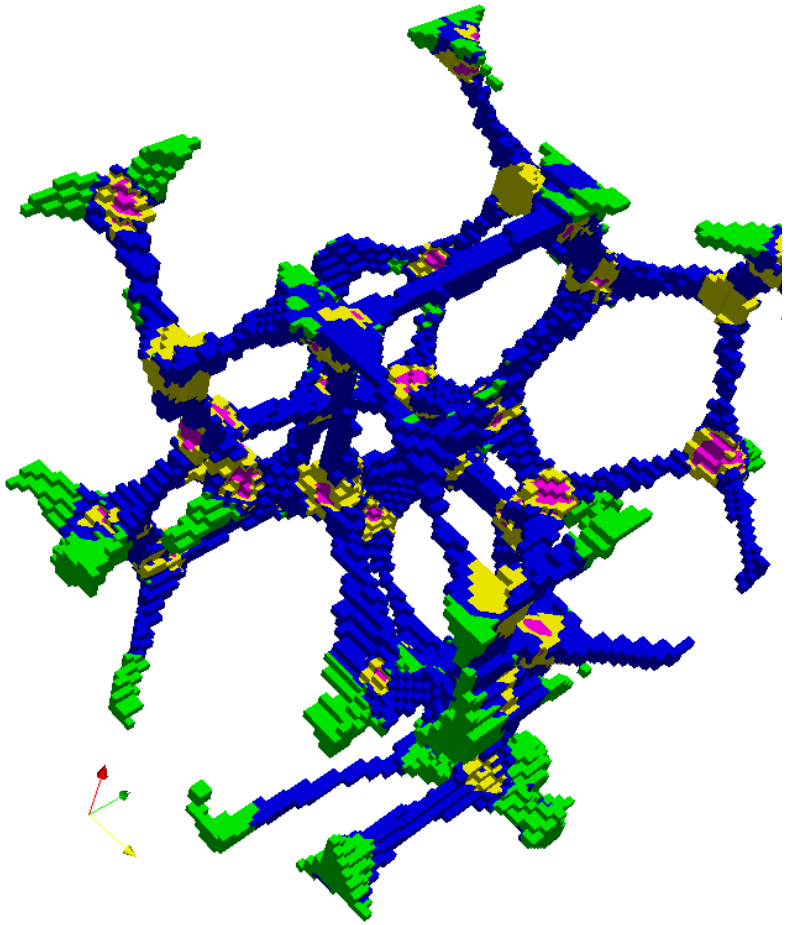
Metric graph reconstruction *[AANJANEYA12]*

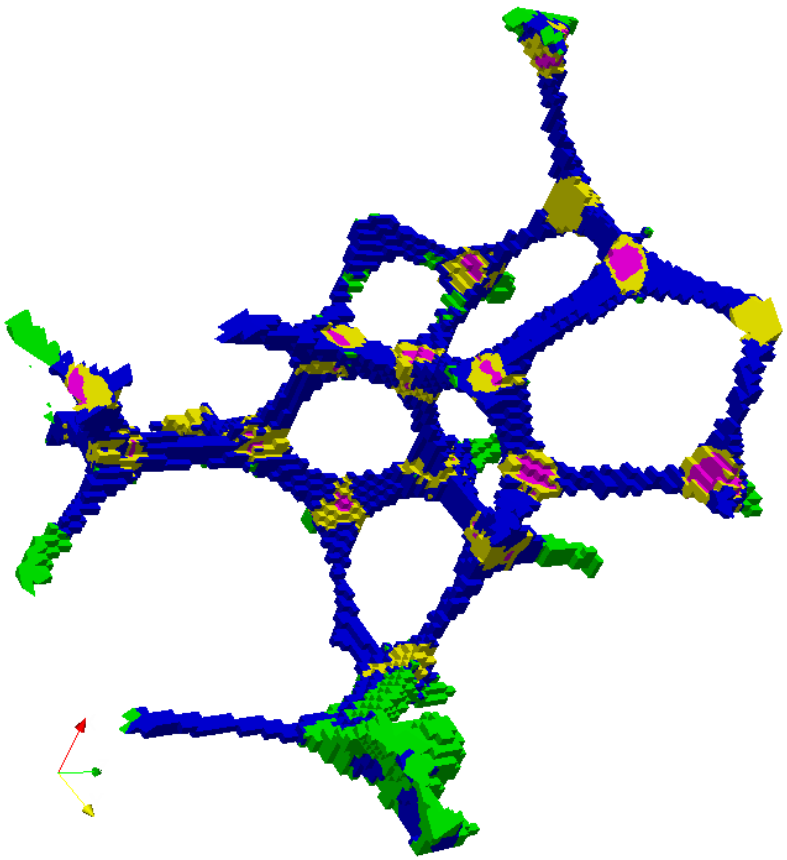


Metric graph reconstruction *[AANJANEYA12]*



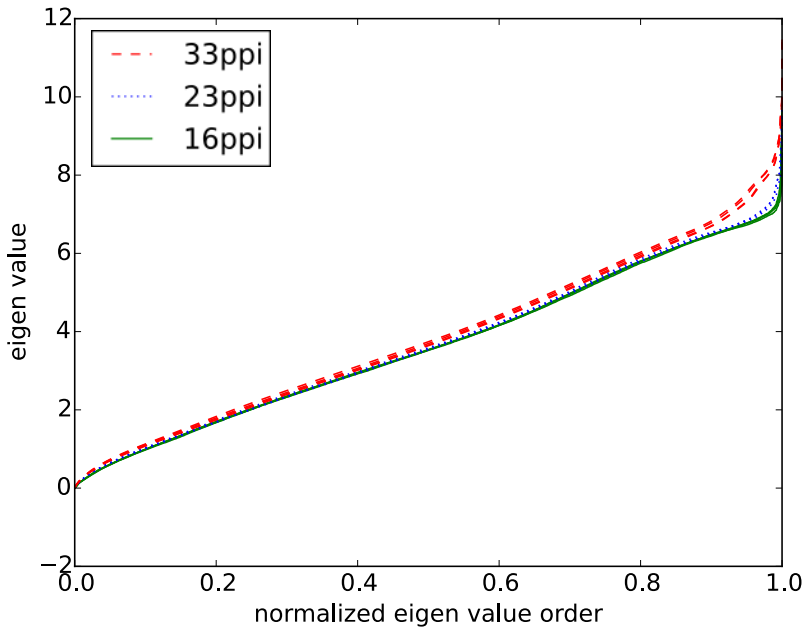
Metric graph reconstruction *[AANJANEYA12]*



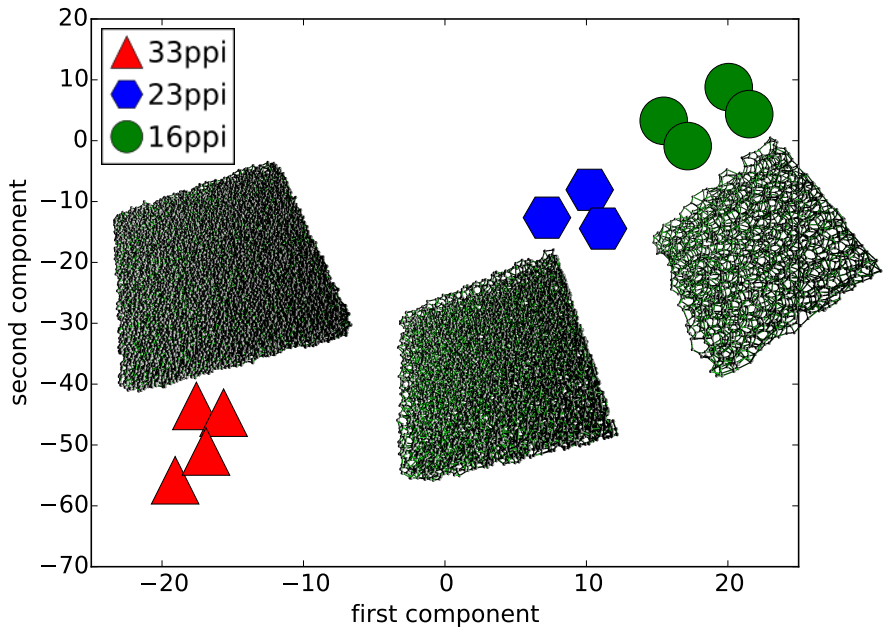


Principal component analysis (PCA) on metric graphs

- 11 samples (4 x 16 ppi, 3 x 23 ppi, 4 x 33 ppi)
- Kirschhoff matrix = degree matrix - adjacency matrix
- PCA on Kirschhoff matrix eigenvalues

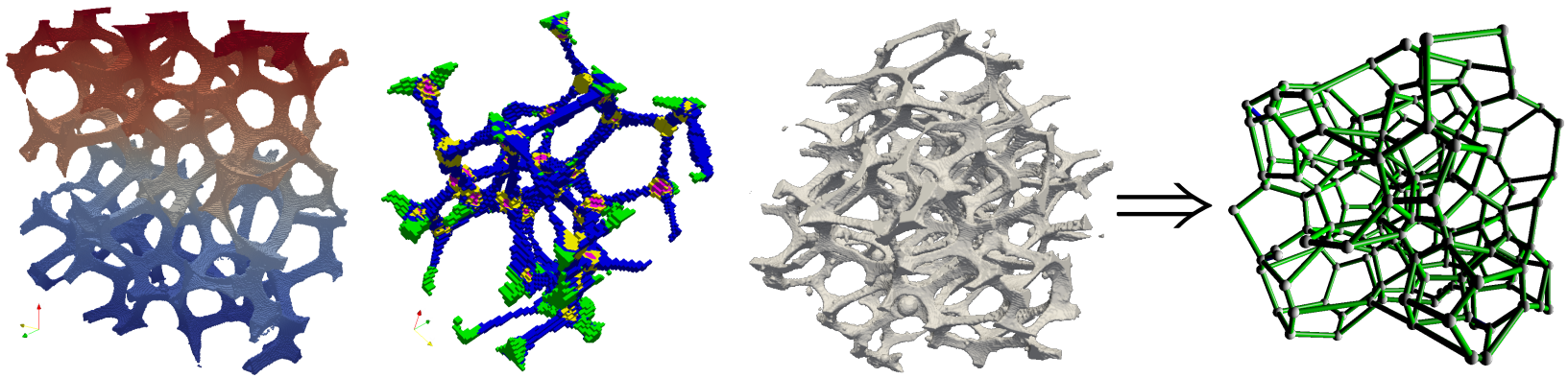


Kirschhoff matrix eigenvalues



PCA projections

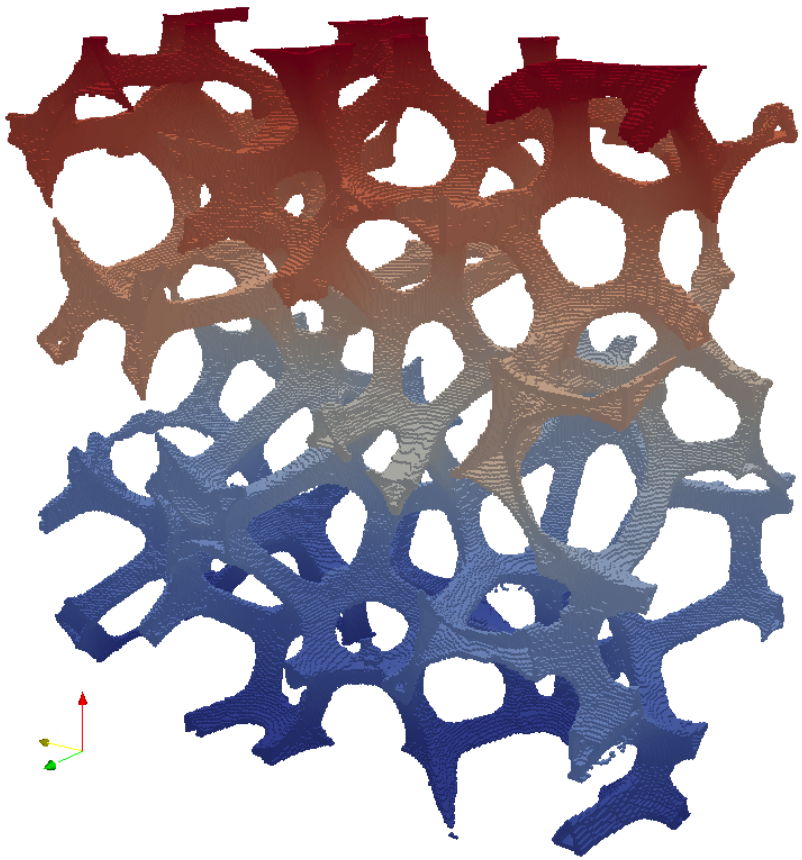
- Convergent toolkit
  - Homotopic thinning preserve topology
  - Proof of convergence of metric graph reconstruction
- Generic C++ implementation
  - Arbitrary ambient and embedded dimensions
  - DEC module included in DGtal
  - Linear algebra solvers from Eigen and ARPACK
  - Open source



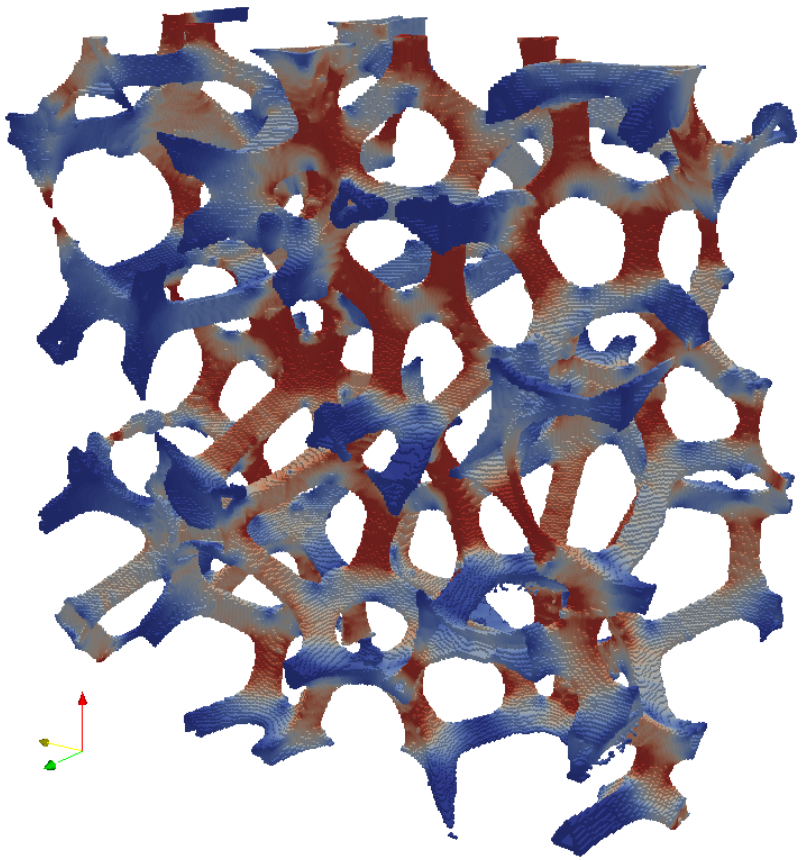
http://dgtal.org

Typical performance : solving heat flow

IO + discretization + problem construction + factorization + resolution



256^3 px $1358000\ \sigma^0$



300 s 1CPU@2.6GHz